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Separable measures and strict topologies on spaces of non-Archimedean valued functions

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Introduction

Let $C_b(X)$ be the space of all bounded continuous functions from a zero-dimensional topological space X to a complete non-Archimedean valued field $\mathbb{K}$. In section 2, we study some of the properties of the space $M(X)$ of all bounded finitely-additive $\mathbb{K}$-valued measures on the algebra of all clopen subsets of X. Among other results, we show that, if m is a τ-additive member of $M(X)$, then the integral with respect to m of an f in $\mathbb{K}^X$, as defined by the author in [5], coincides with the integral defined by Van Rooij in [14] p. 253. In sections 3 and 4 we look at the dual spaces of $C_b(X)$ under the topologies β and β_1 introduced by the author in [6]. In the remaining sections we introduce the topology β_e on $C_b(X)$ which yields as dual space the space $M_s(X)$ of the so called separable members of $M(X)$. It is proved that $M_s(X)$ is also the dual space of $C_b(X)$ under the topology β_u defined in [1] and that β_e and β_u have the same equicontinuous sets in $M_s(X)$. On uniformly bounded equicontinuous subsets of $C_b(X)$, the topologies β_e and β_u and the topology τ_p of simple convergence coincide. It is also proved that the dual space of $(C_b(X), \beta_e)$ is weak*-sequentially complete.

1 Preliminaries

Throughout this paper, $\mathbb{K}$ stands for a complete non-Archimedean valued field whose valuation is not trivial, X a Hausdorff zero-dimensional topological space, $C(X)$ the space of all continuous functions from X to $\mathbb{K}$, $C_b(X)$ the space of all bounded members of $C(X)$ and $C_{rc}(X)$ the space of all f in $C(X)$ whose range in $\mathbb{K}$ is relatively compact. For $f \in \mathbb{K}^X$ and $A \subset X$, we define

$$\|f\| = \sup_{x\in X} |f(x)| \quad and \quad \|f\|_A = \sup_{x\in A} |f(x)|.$$

By a norm (resp. seminorm) on a vector space E over $\mathbb{K}$, we mean a non-Archimedean norm (resp. seminorm). Also, by a locally convex space we mean a non-Archimedean locally convex space over $\mathbb{K}$. For $A \subset X$, χ_A is the $\mathbb{K}$-characteristic function of A. Let $K(X)$ be the algebra of all clopen (i.e. closed and open) subsets of X and let $M(X)$ (see [5]) be the space of all bounded finitely additive $\mathbb{K}$-valued measures on $K(X)$. For $m \in M(X)$ and A a clopen subset of X, we define $|m|(A) = \sup\{|m(B)| : B \in K(X), B \subset A\}$ and $\|m\| = |m|(X)$. For a net $(V_\alpha)_{\alpha\in A}$ of subsets of X, we write $V_\alpha \downarrow B$ if it is decreasing with $\bigcap V_\alpha = B$. We denote by $M_\sigma(X)$ the space of all σ-additive members of $M(X)$ (see [5]), i.e. the set of all $m \in M(X)$ such that $m(A_n) \to 0$ for each sequence (A_n) of clopen sets with $A_n \downarrow \emptyset$. The space $M_\tau(X)$ consists of all τ-additive members of $M(X)$, i.e. those m with $m(V_\alpha) \to 0$ when $V_\alpha \downarrow \emptyset$. If m is σ-additive, then for each sequence (A_n) of clopen subsets of X, which decreases to the empty set, we have that $|m|(A_n) \to 0$ (by [5] ,Theorem 3.2). Also, if m is τ-additive, then for each net (A_δ) of clopen subsets of X, which decreases to the empty set, we have that $|m|(A_\delta) \to 0$ (by [14], p. 249). An m in $M(X)$ is called tight (see [6]) if, for each $\epsilon > 0$ there exists a compact subset D of X such that $|m|(A) \le \epsilon$ for each clopen set A disjoint from D. We will denote by $M_t(X)$ the space of all tight members of $M(X)$.
Next we will recall the definition of the integral of an $f \in \mathbb{K}^X$ with respect to a member m of $M(X)$ (see [5]). An $f \in \mathbb{K}^X$ is called m-integrable if there exists an element of $\mathbb{K}$, which we will denote by $\int f dm$ and will call it the integral of f, such that for each $\epsilon > 0$ there exists a finite clopen partition $\{A_1, \cdots, A_n\}$ of X such that, for each clopen partition $\{B_1, \cdots, B_N\}$ of X,

which is a refinement of $\{A_1, \cdots, A_n\}$, and any choice of $x_i \in B_i$, we have

$$\left| \int f dm - \sum_{i=1}^{N} f(x_i) m(B_i) \right| \leq \epsilon.$$

For an $A \subset X$, we define $\int_A f dm = \int f \chi_A dm$. As is is shown in [5], if $m \in M(X)$, then every $f \in C_{rc}(X)$ is m-integrable and m defines a linear functional $\omega_m : C_{rc}(X) \to \mathbb{K}, \omega_m(f) = \int f dm$, which is continuous with respect to the topology τ_u of uniform convergence. Also, for every $\phi \in (C_{rc}(X), \tau_u)'$ there exists a unique $m \in M(X)$ with $\phi = \omega_m$.
In [14], p. 253, Van Rooij defined as follows the integral of an $f \in \mathbb{K}^X$ with respect to some $m \in M_\tau(X)$. Let

$$N_m : X \to R, N_m(x) = \inf\{|m|(V) : x \in V \in K(X)\}.$$

Then N_m is upper semicontinuous and, for each $\epsilon > 0$, the set $X_\epsilon = \{x \in X : N_m(x) \geq \epsilon\}$ is compact. He defined f to be m-integrable if there exists a sequence (g_n) of $K(X)$-simple functions such that

$$\|f - g_n\|_{N_m} = \sup_{x \in X} |f(x) - g_n(x)| N_m(x) \longrightarrow 0.$$

In this case he defines the integral of f as the $\lim_{n\to\infty} \int g_n dm$, where, for $g = \sum_1^n \alpha_i \chi_{A_i}$, $\int g dm = \sum \alpha_i m(A_i)$. We will denote the above integral by $(VR)\int f dm$.
Next we recall that βo is the locally convex topology on $C_b(X)$ defined by the seminorms $f \mapsto \|gf\|$, where g ranges over the family of all bounded members of $\mathbb{K}^X$ which vanish at infinity. As it is shown in [6], Theorem 3.4, $M_t(X)$ is algebraically isomorphic to the dual space of $(C_b(X), \beta_o)$ via the isomorphism $m \to \phi_m, \phi_m(f) = \int f dm$ for $f \in C_b(X)$. Finally, we will denote by $\beta_o X$ the Banaschewski compactification of X. The $\mathbb{N}$-repletion $\upsilon_o X$ ($\mathbb{N}$ is the set of natural numbers) is the set of all $x \in \beta_o X$ with the following property: If (V_n) is a sequence of clopen neighborhoods of x in $\beta_o X$, then $\bigcap V_n \cap X$ is not empty. For $A \subset X$, we denote by $\bar{A}^{\beta_o X}$ the closure of A in $\beta_o X$.

2 The space M(X)

Theorem 2.1 *If $m \in M(X)$ and $f \in \mathbb{K}^X$, then f is m-integrable iff , for each ϵ 0, there exists a clopen partition $\{A_1, \ldots, A_n\}$ of X such that $|f(x) - f(y)||m|A_i) \leq \epsilon$, for all $i = 1, \ldots, n$, if x, y are in A_i.*

Proof: Assume that f is m-integrable and let $\epsilon > 0$. There exists a clopen partition $\{A_1, \ldots, A_n\}$ of X such that, for each clopen partition $\{B_1, \ldots, B_N\}$ of X, which is a refinement of $\{A_1, \ldots, A_n\}$, and any choice of $y_k \in B_k$, we have $\left|\sum_{k=1}^{N} f(y_k)m(B_k) - \int f dm\right| \leq \epsilon$. Let now $x, y \in A_i$ and let B be a clopen subset A_i. If both x, y are in B, then $|m(B)(f(x) - f(y)| \leq \epsilon$. On the other hand, if $x, y \in A = X \setminus B$, then again $|m(B)(f(x) - f(y)| \leq \epsilon$ since otherwise we would have $|m(A)| = |m(B)|$ (because $|m(A_i)(f(x) - f(y)| \leq \epsilon$) and so $|m(B)(f(x) - f(y)| = |m(A)(f(x) - f(y)| \leq \epsilon$, a contradiction. Finally if (say) $x \in B, y \in A$, then $\epsilon \geq |m(A_i)f(y) - [m(B)f(x) + m(A)f(y)]| = |m(B)(f(x) - f(y)|$. This proves that $|f(x) - f(y)||m|(A_i) \leq \epsilon$. Conversely, assume that the condition is satisfied and let $\{A_1, \ldots, A_n\}$ be a clopen partition of X such that $|f(x) - f(y)||m|(A_i) \leq \epsilon$, for all $i = 1, \ldots, n$, if x, y are in A_i. Let $x_i \in A_i$. If $\{B_1, \ldots, B_N\}$ is a clopen refinement of $\{A_1, \ldots, A_n\}$ and if $y_k \in B_k$, then $\left|\sum_{i=1}^{n} m(A_i)f(x_i) - \sum_{k=1}^{N} m(B_k)f(y_k)\right| \leq \epsilon$. This clearly proves that f is m-integrable and $\left|\int f dm - \sum_{i=1}^{n} m(A_i)f(x_i)\right| \leq \epsilon$.

Theorem 2.2 *Let $m \in M_\tau(X)$ and $f \in \mathbb{K}^X$. If f is m-integrable, then it is integrable in the sense of Van Rooij and $\int f dm = (VR)\int f dm$.*

Proof: Assume that f is m-integrable and let $\epsilon > 0$. In view of the preceding Theorem, there exists a clopen partition $\{A_1, \ldots, A_n\}$ such that $|f(x) - f(y)||m|(A_i) \leq \epsilon$, for all $i = 1, \ldots, n$, if x, y are in A_i. Let $x_i \in A_i$ and $g = \sum_{i=1}^{n} f(x_i)\chi_{A_i}$. Since, for $x \in A_i$, we have $N_m(x) \leq |m|(A_i)$, it follows that $\|f - g\|_{N_m} \leq \epsilon$, which proves tht $(VR)\int f dm$ exists. Moreover

$$\left|(VR)\int f dm - \sum_{i=1}^{n} m(A_i)f(x_i)\right| \leq \|f - g\|_{N_m} \leq \epsilon.$$

Also, as we have seen in the proof of Theorem 2.1, we have $\left|\int f dm - \sum_{i=1}^{n} m(A_i)f(x_i)\right| \leq \epsilon$. It follows that $\left|\int f dm - (VR)\int f dm\right| \leq \epsilon$ and so the two integrals coincide since $\epsilon > 0$ was arbitrary.

Theorem 2.3 $M_\tau(X) = M_t(X)$.

Proof: Let $m \in M_\tau(X)$ and $\epsilon > 0$. The set $X_\epsilon = \{x \in X : N_m(x) \geq \epsilon\}$ is compact. Let A be a clopen subset of X disjoint from X_ϵ. For each $x \in A$, there exists a clopen neighborhood V_x of X with $|m|(V_x) < \epsilon$. Since m is τ-additive and $A \subset \cup_{x \in A} V_x$, we have that $|m|(A) \leq \epsilon$, which proves that m

is tight.
Let now $m \in M(X)$. As in [5], Definition 3.4, we define

$$supp(m) = \bigcap\{V \in K(X) : |m|(X \setminus V) = 0\}.$$

If m is τ-additive, then $m(B) = 0$ if the clopen set B is disjoint from $supp(m)$. For $A \subset X$ we define

$$|m|_*(A) = \sup\{inf_n |m|(A_n) : A_n \in K(X), A_{n+1} \subset A_n, \cap A_n \subset A\}.$$

Theorem 2.4 *Let X be an utraparacompact Hausdorff space and $m \in M(X)$. Then, m is τ-additive iff $supp(m)$ is Lindelöff and $|m|_*(X \setminus supp(m)) = 0$.*

Proof: Assume that m is τ-additive and let (A_n) be a decreasing sequence of clopen sets with $\cap A_n$ disjoint from $supp(m)$. Let $\mathcal{U} = \{V \in K(X) : |m|(X \setminus V) = 0\}$. If $V_1, V_2 \in \mathcal{U}$, then $V_1 \cap V_2 \in \mathcal{U}$ and so $\mathcal{U}$ is downwards directed. The family $\{A_n \cap V : n \in \mathbb{N}, V \in \mathcal{U}\}$ is downwards directed to he empty set. Since m is τ-additive, given $\epsilon > 0$, there exists n and $V \in \mathcal{U}$ with $|m|(A_n \cap V) \leq \epsilon$. Since $|m|(X \setminus V) = 0$, we have that $|m|(A_n) \leq \epsilon$, which proves that $|m|_*(X \setminus supp(m)) = 0$. Next, let $\mathcal{F}$ be a clopen cover of $supp(m)$. The family $\mathcal{V} = \mathcal{F} \cup \{X \setminus supp(m)\}$ is an open cover of X. Because of the paracompactness of X, there is a clopen partition $\{A_i\}_{i \in I}$ of X which is a refinement of $\mathcal{V}$. Let I_1 be the set of those i in I for which A_i is disjoint from $supp(m)$ and let $I_2 = I \setminus I_1$. Then $supp(m) \subset \bigcup_{i \in I_2} A_i$ and, for each $i \in I_2$, A_i is contained in some member of $\mathcal{F}$. For each i, there exists a clopen subset B_i of A_i with $|m(B_i)| \geq |m|(A_i)/2$. For J finite subset of I, set $D_J = \bigcup_{i \in J} B_i$. The set $B = \bigcup_{i \in I} B_i$ is clopen and $B \cap D_J \downarrow \emptyset$. Thus, given $\epsilon > 0$, there exists a finite subset J_o of I such that $|m|(B \setminus D_{J_o}) < \epsilon/2$ and so $|m(B_i)| < \epsilon/2$ if $i \notin J_o$. Thus the set $\{i \in I : |m|(A_i) \geq \epsilon\}$ is contained in J_o. Hence the set $M = \{i \in I : |m|(A_i) \neq 0\}$ is countable and $supp(m) \subset \bigcup_{i \in M} A_i$. Moreover $M \subset I_2$. It is now clear that there exists a countable subfamily of $\mathcal{F}$ covering $supp(m)$, and hence $supp(m)$ is Lindelöff. Conversely, assume that $supp(m)$ is Lindelöff and $|m|_*(X \setminus supp(m)) = 0$. Let (Z_α) be a net of clopen sets decreasing to the empty set. Since the sets $X \setminus Z_\alpha$ cover $supp(m)$, there exists a decreasing sequence (α_n) such that $supp(m) \subset \bigcup_{n=1}^{\infty} X \setminus Z_{\alpha_n}$. Then $Z_{\alpha_n} \downarrow D \subset X \setminus supp(m)$. By our hypothesis, given $\epsilon > 0$, there exists n such that $|m|(Z_{\alpha_n}) < \epsilon$. This clearly proves that m is τ-additive. Hence the result follows.

Theorem 2.5 *If X is ultraparacompact and $m \in M_\sigma(X)$, then m is τ-additive iff $supp(m)$ is Lindelöff and $m(A) = 0$ if A is disjoint from $supp(m)$.*

Proof: The necessity follows from the preceding Theorem. For the sufficiency, it is enough to show that $|m|_*(X \setminus supp(m)) = 0$. So, let (Z_n) be a decreasing sequence of clopen subsets with $Z = \cap Z_n$ disjoint from $supp(m)$. Since X is ultranormal, there exists a clopen subset V of X containing Z and disjoint from $supp(m)$. By our hypothesis, $|m|(V) = 0$. Since $Z_n \bigcap (X \setminus V) \downarrow \emptyset$ and m is σ-additive, there exists n such that $|m|(Z_n) = |m|(Z_n \bigcap (X \setminus V)) \leq \epsilon$, which clearly completes the proof.
The following Theorem is proved in [14], Theorem 7.1.

Theorem 2.6 $M_\sigma(X) = M_\tau(X)$ *iff X is $\mathbb{N}$-replete.*

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Theorem 2.7 *Let $m \in M_\sigma(X)$ and $f \in C_b(X)$. If $f(X)$ has non-measurable cardinal, then f is m-integrable.*

Proof: Let $\bar{m} : K(v_oX) \to \mathbb{K}, \bar{m}(A) = m(A \cap X)$. Then $\bar{m} \in M_\sigma(v_oX)$. Since v_oX is $\mathbb{N}$-replete, we have that $\bar{m} \in M_\tau(v_oX)$. Also, from the hypothesis that $f(X)$ has non-measurable cardinal, it follows that $f(X)$ is $\mathbb{N}$-replete (by [14], Theorem 2.8) and hence there exists a continuous extension $\bar{f}$ of f to all of v_oX. Now $\bar{f}$ is $\bar{m}$-integrable. It follows easily from this that f is m-integrable, which was to be proved.

Theorem 2.8 *Assume that $f(X)$ has non-measurable cardinal for each $f \in C_b(X)$ (e.g. when either X or $\mathbb{K}$ has non-measurable cardinal or when X is $\mathbb{N}$-replete). Let (m_n) be a sequence in $M_\sigma(X)$ such that the limit $\phi(x) = \lim_{n\to\infty} \int f dm_n$ exists for each $f \in C_b(X)$. Then:*
(1) If (Z_n) is a sequence of clopen subsets of X which decreases to the empty set, then $\lim_{n\to\infty} |m_k|(Z_n) = 0$ uniformly for $k = 1, 2, \ldots$.
(2) If $m : K(X) \to \mathbb{K}, m(A) = \lim m_n(A)$, then $m \in M_\sigma(X)$.
(3) If $\mathbb{K}$ is not spherically complete, then $\phi(f) = \int f dm$ for each $f \in C_b(X)$.

Proof: Let $Z_n \downarrow \emptyset$ and let $M = \{f \in C_b(X) : \|f\| \leq 1\}$. Define

$$d : M \times M \to R, d(f,g) = \sup_n \left[\sup_{x \notin Z_n} |f(x) - g(x)|/n \right].$$

Then d is an ultrametric on M. Moreover (M,d) is complete. In fact, let (f_n) be a d-Cauchy sequence in M. For each $x \in X$, the limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists in $\mathbb{K}$. Since $f_n \to f$ uniformly on each $X \setminus Z_k$, it follows that f is continuous. Moreover $d(f_n, f) \to 0$. Next, we observe that, for each $m_o \in M_\sigma(X)$, the map $f \mapsto \int f dm_o$ is d-continuous on M. Indeed, let $\epsilon > 0$. Without loss of generality, we may assume that $\|m_o\| \leq 1$. Since m_o is σ-additive, there exists n_o such that $|m_o|(Z_{n_o}) < \epsilon$. If $d(f,g) < \epsilon/2n_o$, then $|f(x) - g(x)| \leq \epsilon$ if x is not in Z_{no}. It follows easily from this that $|\int(f-g)dm_o| \leq \epsilon$, which proves the continuity of the map $f \mapsto \int f dm_o$. Let now $\delta > 0$. For each $k \in \mathbb{N}$, set

$$M_k = \{g \in M : \left|\int gdm_r - \int gdm_n\right| \leq \delta, \quad for \quad n, r \geq k\}.$$

Each M_k is d-closed and $M = \bigcup_k M_k$. By the Baire category theorem, there exist $k_o \in \mathbb{N}, g_o \in M$ and $\alpha > 0$ suh that $\{g \in M : d(g, g_o) \leq \alpha\} \subset M_{k_o}$. Let now $n_o > 1/\alpha$. Then

$$\{g \in M : \|g - g_o\|_{X \setminus Z_{n_o}} < \alpha\} \subset M_{k_o}.$$

Since eah m_k is σ-additive, there exists $n_1 \geq n_o$ such that $|m_k|(Z_{n_1}) \leq \delta$ for $k = 1, \ldots, k_o$. Let $n > k_o$ and let W be a clopen subset of Z_{n_o}. Let $V = X \setminus Z_{n_o}, g = \chi_W, f = \chi_V, h_1 = fg_o, h_2 = g + h_1$. then $h_1 = h_2 = g_o$ on V and so $h_1, h_2, g \in M_{k_o}$. Thus $|\int gd(m_n - m_{k_o})| \leq \delta$, which proves that $|m_n - m_{k_o}|(Z_{n_o}) \leq \delta$ and so

$$|m_n|(Z_{n_1}) \leq \max\{|m_n - m_{k_o}|(Z_{n_1}), |m_{k_o})|(Z_{n_1})\} \leq \epsilon.$$

Thus $|m_n|(Z_{n_1}) \leq \epsilon$, for all n, which proves (1).
(2) Since $\sup_n |\int f dm_n| < \infty$, for each $f \in C_b(X)$, and since $C_b(X)$ with the uniform topology is a Banach space, it follows that $\sup_{\|f\|\leq 1} |\int f dm_n| < \infty$, which implies that $\sup_n \|m_n\| < \infty$. We get now easily that $m \in M_\sigma(X)$.
(3) Assume that $\mathbb{K}$ is not spherically complete and let τ_u be the topology of uniform cnvergence. The map

$$\phi : (C_b(X), \tau_u) \to \mathbb{K}, \phi(f) = \lim \int f dm_n$$

is continuous. Moreover, $\phi(\chi_A) = m(A)$ for each A in $K(X)$. As it is shown in the proof of Theorem 7.23 in [14], we have that $\phi(f) = \int f dm$ for all $f \in C_b(X)$. This clearly completes the proof.

Theorem 2.9 *$M(X) = M_\sigma(X)$ iff X is $\mathbb{K}$-pseudocompact (equivalently pseudocompact).*

Proof: By [8], Proposition 5.12, X is $\mathbb{K}$-pseudocompact iff $\upsilon_o X = \beta_o X$. Assume that X is not $\mathbb{K}$-pseudocompact and let $x \in \beta_o X \setminus \upsilon_o X$. Let $\mu : K(\beta_o X) \to \mathbb{K}, \mu(A) = 1$, if $x \in A$, and $\mu(A) = 0$, if $x \notin A$. Let $m : K(X) \to \mathbb{K}, m(V) = \mu(\bar{V}^{\beta_o X})$. Then m is in $M(X)$ but not in $M_\sigma(X)$. Indeed, since x is not in $\upsilon_o X$, there exists a decreasing sequence (A_n) of clopen neighborhoods of x in $\beta_o X$ with $\bigcap_{n=1}^{\infty} A_n \cap X = \emptyset$. If $B_n = A_n \cap X$, then $B_n \downarrow \emptyset$ but $m(B_n) = \mu(A_n) = 1$ for all n. Conversely, assume that X is $\mathbb{K}$-pseudocompact and let $m \in M(X)$. If (V_n) is a sequence of clopen subsets of X decreasing to the empty set, then $\overline{V_n}^{\beta_o X} \downarrow \emptyset$. Since $\beta_o X$ is compact, there exists k with $\overline{V_k}^{\beta_o X} = \emptyset$, which implies that $A_n = \emptyset$, for $n \geq k$, and so $\lim m(A_n) = 0$. Thus the result follows.

3 The topologies β and β_1

We recall the definitions of the topologies β and β_1 on $C_b(X)$ which were introduced by the author in [7]. Let Ω be the family of all compact subsets of $\beta_o X$ which are disjoint from X. For $H \in \Omega$, let C_H be the space of all $f \in C_{rc}(X)$ whose the continuous extension f^{β_o}, to all of $\beta_o X$, vanishes on H. We denote by β_H the locally convex topology on $C_b(X)$ generated by the seminorms $p_h, h \in C_H$, where $p_h(f) = \|hf\|$. We define β to be the locally convex inductive limit of the topologies $\beta_H, H \in \Omega$. Replacing Ω by the family Ω_1 of all $\mathbb{K}$-zero subsets of $\beta_o X$ which are disjoint from X, we get the topology β_1. (A $\mathbb{K}$-zero subset of $\beta_o X$ is a set of the form $\{x \in \beta_o X : g(x) = 0\}$ for some $g \in C(\beta_o X)$).

Theorem 3.1 (7, Theorem 4.4) *For $H \in \Omega, \beta_H$ has a base at zero the family of all sets of the form $\bigcap\{f \in C_b(X) : \|f\|_{A_n} \leq |\lambda_n|\}$ where (A_n) is an increasing sequence of clopen subsets of X, with $\overline{A_n}^{\beta_o X}$ disjoint from H, and $\lambda_n \in \mathbb{K}$ with $0 < |\lambda_n| \leq \lambda_{n+1}| \to \infty$.*

Using an argument analogous to the one used in the proof of Theorem 4.1 in [7], we get the following

Theorem 3.2 *An absolutely convex subset V of $C_b(X)$ is a β_H-neighborhood of zero ($H \in \Omega$) iff the following condition is satisfied: For each $d > 0$, there* *exist* $\delta > 0$ *and a clopen subset* A *of* X, *with* $\bar{A}^{\beta_o X}$ *disjoint from* H, *such that* $\{f \in C_b(X) : \|f\| \leq d, \|f\|_A \leq \delta\} \subset V$.

Theorem 3.3 *If* $m \in M_\tau(X)$, *then the map* $\phi_m(f) = \int f dm$ *is* β-*continuous on* $C_b(X)$.

Proof: Let $H \in \Omega$. There exists a decreasing net (B_α) of clopen subsets of $\beta_o X$ with $\bigcap B_\alpha = H$. If $D_\alpha = B_\alpha \cap X$, then $A_\alpha \downarrow \emptyset$. Since m is τ-additive, given $\epsilon > 0$, there exists α_o such that $|m|(D_{\alpha_o}) < 1/d$. If $D = X \setminus D_{\alpha_o}$, then

$$\{f \in C_b(X) : \|f\| \leq d, \|f\|_D \leq 1/\|m\|\} \subset \{f : |\phi_m(f) \leq 1\} = W.$$

In view of the preceding Theorem, W is a β_H-neighborhood of zero, for each $H \in \Omega$, and hence ϕ_m is β-continuous.

Theorem 3.4 *(see [1, Remark 9]) The map* $m \mapsto \phi_m$, *from* $M_\tau(X)$ *to* $(C_b(X), \beta)'$, *is an algebraic isomorhism.*

Proof: Let $\phi \in (C_b(X), \beta)'$. Since β is coarser than τ_u, there exists (by [5, Theorem 2.8]) $m \in M(X)$ such that $\phi(f) = \int f dm$ for each $f \in C_{rc}(X)$. We will show that m is τ-additive. Indeed, let (A_α) be a net of clopen sets with $A_\alpha \downarrow \emptyset$. If $B_\alpha = \overline{A_\alpha}^{\beta_o X}$, then $H = \bigcap B_\alpha \in \Omega$. Let $h \in C_H$ be such that

$$W = \{f \in C_b(X) : \|fh\| \leq 1\} \subset \{f : |\phi(f)| \leq 1\}.$$

Let μ be a nonzero element of $\mathbb{K}$. The set $G = \{x \in \beta_o X : |h^{\beta_o}(x)| \leq |\mu|\}$ is clopen and contains H. There exists α_o with $B_{\alpha_o} \subset G$. If now $\alpha \geq \alpha_o$, then $\mu \chi_A \in W$ and so $|m(A_\alpha)| \leq |\mu|$. This proves that m is τ-additive. Now ϕ and ϕ_m are both β-continuous on $C_b(X)$. Since $C_{rc}(X)$ is β-dense in $C_b(X)$ (by [1, Corollary 8]) and since $\phi = \phi_m$ on $C_{rc}(X)$, it follows that $\phi = \phi_m$ on $C_b(X)$. This clearly completes the proof.

In view of the preceding Theorem, we may consider $M_\tau(X)$ as the dual space of $(C_b(X), \beta)$. Recall that a subset D of $M(X)$ is called uniformly τ-additive if $m(A_\alpha) \to 0$, uniformly for $m \in D$, if $A_\alpha \downarrow \emptyset$. In this case we have that $|m|(A_\alpha) \to 0$ uniformly for $m \in D$ by [7], Theorem 5.4. Analogously, D is called uniformly σ-additive if $m(A_n) \to 0$, uniformly for $m \in D$, if $A_n \downarrow \emptyset$. For D uniformly σ-additive and $A_n \downarrow \emptyset$, we have that $|m|(A_n) \to 0$ uniformly for $m \in D$ by [7], Theorem 5.4. As in [7], Theorem 5.6, we have the following

Theorem 3.5 *A subset D, of the dual space $M_\tau(X)$ of $(C_b(X),\beta)$, is β-equicontinuous iff* $\sup_{m\in H}\|m\|<\infty$ *and D is uniformly τ-additive.*

Theorem 3.6 *Let X be ultraparacompact and let (m_n) be a weak*-Cauchy sequence in the dual space $M_\tau(X)$ of $(C_b(X),\beta)$. Then:*
(1) (m_n) is β-equicontinuous and hence uniformly τ-additive.
(2) *If $m(A)=\lim m_n(A)$ for $A\in K(X)$, then m is τ-additive.*
(3) $\int f dm=\lim\int f dm_n$ for each $f\in C_b(X)$.

Proof: For each n, ϕ_{m_n} is τ-continuous on $C_b(X)$ and the $\lim\phi_{m_n}(f)$ exists for each $f\in C_b(X)$. Since $(C_b(X),\tau_u)$ is a Banach space, it follows that $\sup_n\|\phi_{m_n}\|<\infty$ and so $\sup_n\|m_n\|<\infty$. Let now $A_\alpha\downarrow\emptyset$. By Theorem 2.4, each $supp(m_n)$ is Lindelöff. It follows from this that $D=\overline{\bigcup_n supp(m_n)}$ is Lindelöff and $supp(m)\subset D$. Since $D\subset\bigcup_\alpha X\setminus Z_\alpha$, there a decreasing sequence (α_n) such that $D\subset\bigcup_n X\setminus Z_{\alpha_n}$. If $V_n=Z_{\alpha_n}$, then $V_n\downarrow V\subset X\setminus D$. Since X is ultranormal, there exists a clopen set U containing V and disjoint from D. Now $V_n\bigcap(X\setminus U)\downarrow\emptyset$. Using an argument analogous to the one used in the proof of Theorem 2.8, we get that $|m_k|(V_n\bigcap(X\setminus U))\to 0$ uniformly for $k=1,2,\ldots$. Thus, given $\epsilon>0$, there exists n_o such that $|m_k|(V_{n_o}\bigcap(X\setminus U))<\epsilon$ for all k. Since $|m_k|(U)=0$, it follows that $|m_k|(V_{n_o})\le\epsilon$ for all k. Also $|m|(Z_\alpha)\le\epsilon$ for $\alpha\ge\alpha_{n_o}$ This proves (1) and (2).
(3) Let $f\in C_b(X)$. We may assume that $\|f\|\le 1$ and that $\|m_n\|\le 1$ for all n. Let $\epsilon>0$. There exists a clopen partition $(A_i)_{i\in I}$ of X such that $|f(x)-f(y)|\le\epsilon$ if x,y are in the same A_i. For $J\subset I$ finite, set $V_J=\bigcup_{i\notin J}A_i$. Then $V_J\downarrow\emptyset$. By (1), there exists $J=\{i_1,\ldots,i_n\}$ such that $|m_k|(V_J)\le\epsilon$ for all k. Let $x_k\in A_{i_k}, x_{n+1}\in V_J$. Then $|m_r|(A_{i_k})|f(x)-f(y)|\le\epsilon$, for all r, if $x,y\in A_{i_k}$. Also $|m_r|(V_J)|f(x)-f(y)|\le\epsilon$ if $x,y\in V_J$. Thus $|m|(A_{i_k})|f(x)-f(y)|\le\epsilon$, if $x,y\in A_{i_k}$, and $|m|(V_J)|f(x)-f(y)|\le\epsilon$ if $x,y\in V_J$. As we have seen in the proof of Theorem 2.1, we have

$$\left|\int f dm_k-\sum_{r=1}^n m_k(A_{i_r})f(x_{i_r})-m_k(V_J)f(x_{n+1})\right|\le\epsilon,$$

for all k, and

$$\left|\int f dm-\sum_{r=1}^n m(A_{i_r})f(x_{i_r})-m(V_J)f(x_{n+1})\right|\le\epsilon.$$

It follows from this that $\left|\int f dm - \lim_{k\to\infty}\int f dm_k\right| \leq \epsilon$ and so $\int f dm = \lim_{k\to\infty}\int f dm_k$, which completes the proof.

For a net (f_α) in $\mathbb{K}^X$, we write $|f_\alpha| \downarrow 0$ if, for each $x \in X$, the net $(|f_\alpha(x)|)_\alpha$ decreases to zero.

Theorem 3.7 *(1) If (f_α) is a net in $C_b(X)$ with $|f_\alpha| \downarrow 0$, then $f_\alpha \xrightarrow{\beta} 0$.*
(2) For a sequence (f_n) in $C_b(X)$ with $|f_n| \downarrow 0$, we have $f_\alpha \xrightarrow{\beta_1} 0$.

Proof: (1) Without loss of generality, we may assume that $\|f_\alpha\| \leq 1$ for all α. Let W be a convex β-neighborhood of zero. There exists $\epsilon > 0$ such that $W_1 = \{f \in C_b(X) : \|f\| \leq \epsilon\} \subset W$. Let $D_\alpha = \{x : |f_\alpha(x) \geq \epsilon\}$. Then $Q = \bigcap_\alpha \overline{D_\alpha}^{\beta_o X} \in \Omega$. Let $h \in C_Q$ be such that $W_2 = \{f \in C_b(X) : \|hf\| \leq 1\} \subset W$. Let $M = \{x \in X : |h(x)| \geq 1\}$. There exists α_o such that $M \subset X \setminus D_{\alpha_o}$. Let $\alpha \geq \alpha_o, g_1 = f_\alpha \chi_M, g_2 = f_\alpha - g_1$. Then $g_1 \in W_1, g_2 \in W_2$ and $f_\alpha = g_1 + g_2 \in W$.
(2) The proof is analogous to that of (1) since, for $B_n = \{x : |f_n(x)| \geq \epsilon\}$, we have that $\bigcap \overline{B_n}^{\beta_o X} \in \Omega_1$.

Let us say that a subset V of $C_b(X)$ is solid if $g \in V, f \in C_b(X), |f| \leq |g|$ imply that $f \in V$. A locally convex topology on $C_b(X)$ is called locally solid if it has a base at zero consisting of solid sets.

Theorem 3.8 *The topologies β and β_1 are locally solid.*

Proof: Let W be a convex β-neighborhood of zero. For each $Q \in \Omega$, there exists $h_Q \in C_Q$ such that $W_Q = \{f \in C_b(X) : \|h_Q f\| \leq 1\} \subset W$. The set $V = \bigcup_{Q\in\Omega} W_Q$ is solid and its absolutely convex hull $co(V)$ is a β-neighborhood of zero contained in W. We will show that $co(V)$ is solid. In fact, let $g \in co(V)$ and $f \in C_b(X)$ with $|f| \leq |g|$. There are $g_1, \ldots, g_n \in V$ such that $g = \sum_{k=1}^n g_k$. Then $|f| \leq \max\{|g_1|, \ldots, |g_n|\}$. By [9, Proposition 2.4], there are $f_1, \ldots, f_n \in C(X), |f_k| \leq |g_k|, f = \sum_{k=1}^n f_k$. Since V is solid, each f_k is in V and so $f \in co(V)$. This proves the result for β. The proof for β_1 is analogous.

For a β-equicontinuous subset H of $M_\tau X$, let

$$N_H : X \to R, \quad N_H(x) = \inf\{\sup_{m\in H} |m|(V) : x \in V \in K(X)\}.$$

The proof of the following Theorem is analogous to the proof of Theorem 7.6 in [14].

Theorem 3.9 *If H is a β-equicontinuous subset of $M_\tau(X)$, then :*
(1) N_H is upper semicontinuous.
(2) For every $\epsilon > 0$, the set $X_{H,\epsilon} = \{x : N_H(x) \geq \epsilon\}$ is compact.

Proof: (1) It suffices to show that, for each $\theta > 0$, the set $W = \{x \in X : N_H(x) < \theta\}$ is open. So let $x \in W$. There exists a clopen neighborhood V of x such that $\sup_{m\in H} |m|(V) < \theta\}$. Then $V \subset W$.
(2) Let $\mathcal{F}$ be a clopen cover of $X_{H,\epsilon}$. We may assume that $\mathcal{F}$ is closed under finite unions. The set $M = X \setminus X_{H,\epsilon}$ is a union of clopen sets. The family

$$\mathcal{V} = \{X \setminus (U \cup V) : U \in \mathcal{F}, V \in K(X), V \subset M\}$$

is downwards directed to the empty set. Since H is uniformly τ-additive, there exists $U \in \mathcal{F}$ and $V \subset M$ with $|m|(X \setminus (U \cup V)) < \epsilon/2$ for all $m \in H$. Thus $X_{H,\epsilon} \subset U \cup V$ and so $X_{H,\epsilon} \subset U$. This completes the proof.

We know that $M_\tau(X)$ is the common dual space for both topologies β and β_o on $C_b(X)$. Moreover β is finer than β_o.

Theorem 3.10 *A subset H of $M_\tau(X)$ is β-equicontinuous iff it is β_o-equicontinuous.*

Proof: Assume that H is β-equicontinuous. To show that H is β_o-equicontinuous, it suffices to show that H is tight. Since $sup_{m\in H}\|m\| < \infty$, we only need to show that, for each $\epsilon > 0$, there exists a compact subset Y of X such that $|m|(V) < \epsilon$, for every $m \in H$, if V is disjoint from Y. Given $\epsilon > 0$, take $Y = X_{H,\epsilon}$. Let V be a clopen set disjoint from Y. For eah $x \in V$ there exists a clopen neighborhood V_x of x such that $\sup_{m\in H} |m|(V_x) < \epsilon$. Let $m \in H$. Since m is τ-additive and $V \subset \bigcup_{x\in V} V_x$, we have that $|m|(V) \leq \epsilon$. This clearly completes the proof.

Since β_o is a polar topology, we get as a Corollary the following

Theorem 3.11 *β_o is the polar topology associated with β.*

For a continuous map $\phi : X \to Y$, where X, Y are zero dimensional Hausdorff spaces, let

$$T_\phi : C_b(Y) \longrightarrow C_b(X), \quad f \mapsto f \circ \phi.$$

Theorem 3.12 *If $\phi : X \to Y$ is continuous, then T_ϕ is $\beta - \beta, \beta_o - \beta_o$ and $\beta_1 - \beta_1$ continuous.*

Proof: Let W be a convex β-neighborhood of zero in $C_b(X)$ and $V = T_\phi^{-1}(W)$. Let Q be a compact subset of $\beta_o Y \setminus Y$ and let $\phi^{\beta_o} : \beta_o X \to \beta_o Y$ be the continuous extension of ϕ. There exists a decreasing net $(W_\alpha)_{\alpha \in I}$ of clopen subsets of $\beta_o Y$ with $\bigcap W_\alpha = Q$. Let $V_\alpha = (\phi^{\beta_o})^{-1}(W_\alpha), H = \bigcap V_\alpha$. Then H is disjoint from X. Since W is a β-neighborhood of zero in $C_b(X)$, given $d > 0$ there exist $\epsilon > 0$ and a clopen subset A of X, with $\bar{A}^{\beta_o X} \bigcap H = \emptyset$, such that

$$\{f \in C_b(X) : \|f\| \leq d, \|f\|_A \leq \epsilon\} \subset W.$$

There exists α such that $\bar{A}^{\beta_o X} \subset \beta_o X \setminus V_\alpha$. Let $B = Y \setminus W_\alpha \bigcap Y$. Then $\bar{B}^{\beta_o Y}$ is disjoint from Q and

$$\{f \in C_b(Y) : \|f\| \leq d, \|f\|_B \leq \epsilon\} \subset V.$$

It follows now from Theorem 3.2 that V is a β_Q-neighborhood of zero in $C_b(Y)$ and so V is a β-neighborhood of zero. Hence T_ϕ is $\beta - \beta$ continuous. The proof for the case of β_1 is analogous. Finally, since a continuous linear map $T : E \to F$, between two locally convex spaces, is also continuous when we consider on E, F the corresponding polar topologies, it follows that T_ϕ is $\beta_o - \beta_o$ continuous.

4 The dual space of $(C_b(X), \beta_1)$

Theorem 4.1 *If $\phi \in (C_b(X), \beta_1)'$, then there exists a unique $m \in M_\sigma(X)$ such that $\phi(f) = \int f dm$ for all $f \in C_{rc}(X)$.*

Proof: Since β_1 is coarser than τ_u, there exists a unique $m \in M(X)$ such that $\phi(f) = \int f dm$ for all $f \in C_{rc}(X)$. We need to show that m is σ-additive. So, let (A_n) be a sequence of clopen sets which decreases to the empty set. Then $\chi_{A_n} \to 0$, with respect to the topology β_1, by Theorem 3.7, and so $m(A_n) = \phi(\chi_{A_n}) \to 0$.

For a τ_u-continuous linear functional ϕ on $C_b(X)$, we define

$$\|\phi\| = \sup\{|\phi(f)|/\|f\| : f \in C_b(X), f \neq 0\}.$$

Theorem 4.2 *Let $m \in M(X)$ be such that every $f \in C_b(X)$ is m-integrable. Then the linear fuctional $\phi_m : C_b(X) \longrightarrow \mathbb{K}, \phi_m(f) = \int f dm$, is β_1-continuous iff m is σ-additive.*

Proof: If ϕ_m is β_1-continuous, then m is σ-additive by the preceding Theorem. Converesely, assume that m is σ-additive. We may assume that $\|m\| \leq 1$. Let $Q \in \Omega_1$. There exists a decreasing sequence (A_n) of clopen subsets of X such that $\bigcap \overline{A_n}^{\beta_o X} = Q$. Let $d > 0$ and choose n_o such that $|m|(A_n) < 1/d$ if $n \geq n_o$. If $A = X \setminus A_{n_o}$, then $\bar{A}^{\beta_o X}$ is disjoint from Q and

$$\{f \in C_b(X) : \|f\| \leq d, \|f\|_A \leq 1\} \subset \{f : |\phi_m(f)| \leq 1\} = V.$$

Hence the result follows from Theorem 3.2.

Recall that if either X or $\mathbb{K}$ has non-measurable cardinal or when X is $\mathbb{N}$-replete, then $f(X)$ has non-measurable cardinal, for each $f \in C_b(X)$, and so each $f \in C_b(X)$ is m-integrable for each $m \in M_\sigma(X)$.

Theorem 4.3 *Let $\mathbb{K}$ be non-spherically complete and assume that $f(X)$ has non-measurable cardinal, for each $f \in C_b(X)$. Then, the map $m \mapsto \phi_m$, from $M_\sigma(X)$ to $(C_b(X), \beta_1)'$, is an algebraic isomorphism.*

Proof: Let $\phi \in (C_b(X), \beta_1)'$. By Theorem 4.1, there exists $m \in M_\sigma(X)$ such that $\phi(f) = \phi_m(f)$ for each $f \in C_{rc}(X)$. Since $\phi(\chi_A) = \phi_m(\chi_A)$, for each $A \in K(X)$, we have that $\phi = \phi_m$ as it was shown in the proof of Theorem 7.23 in [14]. It is now clear that the result follows.

Under the hypotheses of the preceding Theorem, $M_\sigma(X)$ may be identified with the dual space of $(C_b(X), \beta_1)$.

Theorem 4.4 *Let $\mathbb{K}$ be non-spherically complete and assume that $f(X)$ has non-measurable cardinal, for each $f \in C_b(X)$. Then, a subset H of $M_\sigma(X)$ is β_1-equicontinuous iff it is uniformly σ-additive and $\sup_{m \in H} \|m\| < \infty$.*

Proof: The proof is analogous to the one of Theorem 5.6 in [7].

5 The topology β_e

In this section we will introduce a locally convex topology on $C_b(X)$ which will yield as dual space the space of the so called separable members of $M(X)$.

For d a bounded continuous ultra-pseudometric on X, we will denote by X_d the quotient space $X/\sim$, where $\sim$ is the equivalence relation on X defined by $x \sim y$ iff $d(x,y) = 0$. If $\tilde{x}_d$ is the equivalence class of x, then X_d becomes an ultrametric space under the metric $\tilde{d}(\tilde{x}_d, \tilde{y}_d) = d(x,y)$. Let $\pi_d : X \to X_d$ be the quotient map. Since π_d is continuous, we get a linear map

$$T_d : C_b(X_d) \longrightarrow C_b(X), \quad T_d f = f \circ \pi_d.$$

Each $g \in C_b(X)$ defines a bounded continuous ultra-pseudometric $d = d_g$ on X by $d(x,y) = |g(x) - g(y)|$. The function $f(\tilde{x}_d) = g(x)$ is well defined on X_d and $T_d f = g$. Thus $C_b(X)$ is the union of all $T_d(C_b(X_d))$ as d ranges over the family of all bounded continuous ultra-pseudometrics on X. We define $(C_b(X), \beta_e)$ to be the locally convex inductive limit of the spaces $(C_b(X_d), \beta)$, with respect to the linear maps T_d, where d ranges over the family of all bounded continuous ultra-pseudometrics on X.

We will recall the definition of the topology β_u which was given in [1]. Let Ω_u be the family of all $Q \in \Omega$ with the following property: There exists a clopen partition $(A_i)_i$ of X with $\overline{A_i}^{\beta_o X}$ disjoint from Q for all i. Then β_u is the inductive limit of the topologies $\beta_Q, Q \in \Omega_u$.

Theorem 5.1 *Let $\phi : X \to Y$ be continuous, where X, Y are zero-dimensional Hausdorff spaces. Then the map*

$$T : C_b(Y) \to C_b(X), f \mapsto f \circ \phi,$$

is $\beta_u - \beta_u$ continuous.

Proof: Let W be convex β_u-neighborhood of zero in $C_b(X)$ and let $r > 0$. Let $(A_i)_{i \in I}$ be a clopen partition of Y. If $B_i = \phi^{-1}(A_i)$, then (B_i) is a clopen partition of X. Since W is a β_u-neighborhood of zero, there exist $\epsilon > 0$ and a clopen subset B of X, with $\bar{B}^{\beta_o X}$ contained in $\bigcup_i \bar{B}_i^{\beta_o X}$, such that

$$D = \{g \in C_b(X) : \|g\| \leq r, \|g\|_B \leq \epsilon\} \subset W.$$

Let J be a finite subset of I such that $\bar{B}^{\beta_o X} \subset \bigcup_{i \in J} \bar{B}_i^{\beta_o X}$. If $A = \bigcup_{i \in J} A_i$, then $\bar{A}^{\beta_o X} \subset \bigcup_{i \in I} \bar{A}_i^{\beta_o Y}$ and

$$\{f \in C_b(Y) : \|f\| \leq r, \|f\|_A \leq \epsilon\} \subset T^{-1}(W).$$

This clearly proves that $T^{-1}(W)$ is a β_u-neighborhood of zero in $C_b(Y)$ and so T is $\beta_u - \beta_u$ continuous.

Theorem 5.2 $\beta_u \leq \beta_e \leq \beta_1$.

Proof: Since for ultraparacompact spaces (and hence for ultrametrizable spaces) we have $\beta = \beta_u$ ([1], Theorem 3) and since, for each bounded continuous ultra-pseudometric d on X, the map T_d is $\beta_u - \beta_u$ continuous, it follows that $\beta_u \leq \beta_e$. Assume now that there exists a convex β_e-neighborhood W of zero which is not a β_1-neighborhood of zero. Then, there exists $Q \in \Omega_1$ and $r > 0$ such that, for each clopen subset A of X, with $\bar{A}^{\beta_o X} \bigcap Q = \emptyset$, and each $\epsilon > 0$, there exists $f \in C_b(X)$, with $\|f\| \leq r, \|f|_A \leq \epsilon$, which is not in W. Let $h \in C_{rc}(X)$ be such that $Q = \{x \in \beta_o X : h^{\beta_o}(x) = 0\}$. For each n set $A_n = \{x \in X : |h(x)| \geq 1/n\}$. There exists $f_n \in C_b(X), \|f_n| \leq r, \|f_n|_{A_n} \leq 1/n, f_n \notin W$. Let $f_o = h$ and define

$$d : X \times X \to R, d(x,y) = \sup\{|f_n(x) - f_n(y)| : n = 0, 1, \ldots\}.$$

Then d is a bounded continuous ultra-pseudometric on X and so $T_d^{-1}(W)$ is a β-neighborhood of zero in $C_b(X_d)$. Let $\pi_d^{\beta_o} : \beta_o X \to \beta_o Y$ be the continuous extension of π_d. Then $H = \pi_d^{\beta_o}(Q)$ is disjoint from X_d. Indeed, assume that there exists $x \in Q$ and $a \in X$ with $\pi_d^{\beta_o}(x) = \pi_d^{\beta_o}(a)$. Let (x_δ) be a net in X converging to x. Then $\pi_d(x_\delta) \to \pi_d^{\beta_o}(x) = \pi_d(a)$ and so $d(x_\delta, a) \to 0$. Since $f_o(a) \neq 0$, there exists δ_o such that $|f_o(x_\delta) - f_o(a)| < |f_o(a)|$, for $\delta \geq \delta_o$, and so $|f_o(x_\delta)| = |f_o(a)|$ for large δ, which is a contradiction since $h(x_\delta) \to h^{\beta_o}(x) = 0$. Thus H is disjoint from X_d. Since $T_d^{-1}(W)$ is a β_H-neighborhood of zero, there exist $\epsilon > 0$ and a clopen subet A of X_d, with $\bar{A}^{\beta_o X_d} \bigcap H = \emptyset$, such that

$$\{f \in C_b(X_d) : \|f\| \leq r, \|f\|_A \leq \epsilon\} \subset T_d^{-1}(W).$$

Let $B = \pi_d^{-1}(A)$. since

$$\pi_d^{\beta_o}(\bar{B}^{\beta_o X}) \subset \bar{A}^{\beta_o X_d} \subset \bar{B}^{\beta_o X_d} \setminus H,$$

we have that $\bar{B}^{\beta_o X} \subset \beta_o X \setminus Q$ and so $\bar{B}^{\beta_o X} \subset \bigcup_n \overline{A_n}^{\beta_o X}$. Let $n \geq 1/\epsilon$ be such that $\bar{B}^{\beta_o X} \subset \overline{A_n}^{\beta_o X}$. The function $g : X_d \to \mathbb{K}, g(\tilde{x}_d) = f_n(x)$, is well defined and continuous. Moreover $\|g\| \leq r$ and $\|g\|_A = \sup_{x \in B} |f_n(x)| \leq \|f_n\|_{A_n} \leq \epsilon$ and thus $g \in T_d^{-1}(W)$, which implies that $f_n = T_d g \in W$, a contradiction. This clearly completes the proof.

Theorem 5.3 *For each β_e-neighborhood W of zero in $C_b(X)$ and each $f \in C_b(X)$, there exists a $K(X)$-simple function g, with $g(X) \subset f(X)$, such that $f - g \in W$.*

Proof: We may assume that W is convex. Since β_e is coarser than the topology of uniform convergence, there exists $\epsilon > 0$ such that $W_1 = \{g \in C_b(X) : \|g\| \leq \epsilon\} \subset W$. Consider the equivalence relation $\sim$ on X defined by $x \sim y$ iff $|f(x) - f(y)| \leq \epsilon$. Let $(A_i)_{i \in I}$ be the corresponding clopen partition of X. If $f_i = \chi_{A_i}$ and if $x_i \in A_i$, then $f^\star = \sum_{i \in I} f(x_i) f_i$ is in $C_b(X)$ and $f - f^\star \in W$. Define

$$d(x,y) = \max\{|f(x) - f(y)|, \sup_i |f_i(x) - f_i(y)|\}.$$

Then d is a bounded continuous ultra-pseudometric on X and so $V = T_d^{-1}(W)$ is a β-neighborhood of zero in $C_b(X_d)$. Let $K = \beta_o X \setminus \bigcup_{i \in I} \overline{A_i}^{\beta_o X}$ and $Q = \pi_d^{\beta_o}(K)$. Then Q is compact and disjoint from X_d. Since V is a β-neighborhood of zero in $C_b(X_d)$, there exists a clopen subset A of X_d, whose closure in $\beta_o X_d$ is disjoint from Q, and $\epsilon_1 > 0$ such that

$$\{g \in C_b(X_d) : \|g\| \leq \|f\|, \|g\|_A \leq \epsilon_1\} \subset V.$$

Let $B = \pi_d^{-1}(A)$. Then $\bar{B} \subset \bigcup_i \overline{A_i}^{\beta_o X}$ and so there are $i_1, i_2, \ldots, i_n \in I$ such that $B \subset \bigcup_{k=1}^n A_{i_k}$. Let $g = \sum_{k=1}^n f(x_{i_k}) f_{i_k}$ and $g_1 = f^\star - g$. The function $\bar{g}_1 : X_d \to \mathbb{K}, \bar{g}_1(\tilde{x}_d) = g_1(x)$, is well defined and continuous. Moreover, $\|\bar{g}_1\| \leq \|f\|$ ad $\bar{g}_1 = 0$ on A, which implies that $\bar{g}_1 \in V$ and so $g_1 = T_d \bar{g}_1 \in W$. Finally, $f - g = (f - f^\star) + g_1 \in W$, which was to be proved.

Theorem 5.4 *If τ_p is the topology of simple convergence, then $\tau_p = \beta_{se}$ on each uniformly bounded equicontinuous subset H of $C_b(X)$.*

Proof: Since $co(H)$ is also uniformly bounded and equicontinuous, we may assume that H is absolutely convex. Let W be a convex β_e-neighborhood of zero. Since β_e is coarser than the topology of simple convergence, there exists $\epsilon > 0$ such that

$$W_1 = \{f \in C_b(X) : \|f\| \leq \epsilon\} \subset W.$$

Define $d(x,y) = \sup_{f \in H} |f(x) - f(y)|$. Then d is a bounded continuous ultra-pseudometric on X. Let $(A_i)_i$ be the clopen partition of X corresponding to the equivalence relation $x \sim y$ iff $d(x,y) \leq \epsilon$. The set $K = \beta_o X \setminus \bigcup_i \bar{A}_i^{\beta_o X}$ is in $\Omega(X)$. If $Q = \pi_d^{\beta_o}(K)$, then Q is compact and disjoint from $\beta_o X_d$. As $V = T_d^{-1}(W)$ is a β-neghborhood of zero, there exist a clopen subset B of X_d, with $\bar{B}^{\beta_o X_d}$ disjoint from Q, and $\epsilon_1 > 0$ such that

$$\{g \in C_b(X_d) : \|g\| \leq r, \|g\|_B \leq \epsilon_1\} \subset V,$$

where $r > \sup_{f\in H} \|f\|$. If $A = \pi_d^{-1}(B)$, then $\bar{A}^{\beta_o X} \subset \bigcup_{i\in I} \bar{A}_i^{\beta_o X}$. Let J be a finite subset of I such that $\bar{A}^{\beta_o X} \subset \bigcup_{i\in J} \bar{A}_i^{\beta_o X}$. Then $W_2 = \{f \in H : |f(x_i| \leq \epsilon_1, i \in J\} \subset W$. Indeed, let $f \in W_2$ and $f^* = \sum_{i\in I} f(x_i)\chi_{A_i}$. The function $h : X_d \to \mathbb{K}, h(\tilde{x}_d) = f^*(x)$, is well defined and continuous. If $\tilde{x}_d \in B$, then $x \in A$ and so $x \in A_i$, for some $i \in J$, which implies that $|f^*(x)| = |f(x_i)| \leq \epsilon_1$. Thus $h \in V$ and so $f^* = T_d h \in W$. Also, $f - f^* \in W_1 \subset W$, which implies that $f \in W$. This proves that the topology induced on H by β_e is coarser than the one induced by τ_p and hence τ_p and β_e induce the same topology on H which was to be proved.

Theorem 5.5 *Let X, Y be Hausdorff zero-dimensional spaces and $\phi : X \to Y$ be a continuous map. Then the map*

$$S_\phi : C_b(Y) \to C_b(X), f \mapsto f \circ \phi,$$

is $\beta_e - \beta_e$ continuous.

Proof: It suffices to show that , for each bounded continuous ultra-pseudometric d on Y, the composed map

$$S_\phi \circ T_d : (C_b(Y_d), \beta) \longrightarrow (C_b(X), \beta_e)$$

is continuous. Given such a d, define d_1 on $X \times X$ by $d_1(x,y) = d(\phi(x), \phi(y))$. Let $\psi : X_{d_1} \to Y_d, \tilde{x}_{d_1} \mapsto \tilde{\phi}(x)_d$. Then ψ is well defined and continuous. Let $S_\psi : C_b(Y_d) \to C_b(X_{d_1})$ be the induced linear map. Then $T_{d_1} \circ S_\psi = S_\phi \circ T_d$. Since T_{d_1} is $\beta - \beta_e$continuous and S_ψ $\beta - \beta$-continuous, it follows that $S_\phi \circ T_d$ is $\beta - \beta_e$ scontinuous. Hence the result follows.

6 The dual space of $(C_b(X), \beta_e)$

Using Theoremsa 5.2 and 5.3, we get the following

Theorem 6.1 *There exist a subspace $M_e(X)$ and an algebraic isomorphism*

$$S_e : M_e(X) :\to (C_b(X), \beta_e)', m \mapsto u_m,$$

such that $u_m(f) = \int f dm$ for all $f \in C_{\tau c}(X)$.

We will show that $M_e(X)$ consists of the so called separable members of $M(X)$.

Let $m \in M(X)$ and let d be a bounded continuous ultra-pseudometric on X. For each d-clopen subset A of X, we define $|m|_d(A) = \sup |m|(B)$, where the supremum is taken over the family of all d-clopen sets B containing A. Also, for $D \subset X$, we define $|m|_d^\star(A) = \inf \sup_n |m|_d(A_n)$, where the infimum is taken over the family of all sequences (A_n) of d-clopen sets with $A \subset \bigcup_n A_n$.

Theorem 6.2 *If (X, d) is an ultrametric space and $m \in M_\tau(X)$, then there exists a d-closed, d-separable subset G of X such that $|m|_d^\star(X \setminus G) = 0$.*

Given $\epsilon > 0$, consider the family of all sets $N(Y, \epsilon)$, where Y is a finite subset of X and $N(Y, \epsilon) = \{x : d(x, Y) \leq \epsilon\}$. Each $N(Y, \epsilon)$ is d-clopen. If Y_1, Y_2 are finite subsets of X and $Y = Y_1 \bigcup Y_2$, then $N(Y, \epsilon) = N(Y_1, \epsilon) \bigcup N(Y_2, \epsilon)$. Moreover $X = \bigcup \{N(Y, \epsilon) : Y \subset X, Y finite\}$. If $D_Y = X \setminus N(Y, \epsilon)$, then (D_Y) is downwards directed to the emptry set. Since m is τ-additive, given $\epsilon_1 > 0$, there exists a finite subset Y of X with $|m|_d(D_Y) < \epsilon_1$. Now, for positive integers n, k, there exists a finite set $Y_{n,k}$ with $|m|_d(D(Y_{n,k}, 1/k)) < 1/n$. Let

$$D_n = \bigcup_k (X \setminus N(Y_{n,k}, 1/k)), \quad M = \bigcup_n (X \setminus D_n)$$

and $\bar{M} = G$. Then $X \setminus G \subset \bigcap_n D_n$. Since $|m|_d(X \setminus N(Y_{n,k}, 1/k)) < 1/n$, it follows that $|m|_d^\star(X \setminus G) = 0$. The set G is d-closed. Also G is d-separable. In fact, let $\epsilon > 0$ and $x \in G$. There exists $y \in M$ with $d(x, y) < \epsilon$. Choose $k > 1/\epsilon$ and let n be such that $y \in N(Y_{n,k}, 1/k))$. There exists $z \in Y_{n,k}$ with $d(z, y) \leq 1/k < \epsilon$ and so $d(x, z) < \epsilon$. The set $M = \bigcup_{n,k} Y_{n,k}$ is countable and $G \subset \bar{Y}$. Since $\bar{Y}$ is a separable metric spasce, its subspace G is also separable. Hence the result follows.

Definition 6.3 *An element m of $M_\sigma(X)$ is called separable if, for each bounded continuous ultra-pseudometric d on X, there exists a d-closed, d-separable subset G of X such that $|m|_d^\star(X \setminus G) = 0$. We will denote by $M_s(X)$ the space of all separable members of $M_\sigma(X)$.*

In the following Theorem we show that $M_s(X) = M_e(X)$.

Theorem 6.4 *$M_s(X)$ is algebraically isomorphic to the dual space of $(C_b(X), \beta_e)$.*

Proof: Let $\phi \in (C_b(X), \beta_e)'$. Since $\beta_e \le \beta_1$, there exists $m \in M_\sigma(X)$ such that $\phi(f) = \int f dm$ for all $f \in C_{rc}(X)$. We will show that m is separable. Let d be a bounded continuous ultra-pseudometric on X. Since the map

$$T_d : (C_b(X_d), \beta) \to (C_b(X), \beta_e)$$

is continuous, $T_d'\phi$ is β-continuous on $C_b(X_d)$. Thus, there exists $\mu \in M_\tau(X_d)$ such that $\langle T_d f, m\rangle = \langle f, \mu\rangle$ for all $f \in C_{rc}(X_d)$. By Theorem 6.2, there exists a $\tilde{d}$-closed, $\tilde{d}$-separable subset Z of X_d such that $|\mu|_{\tilde{d}}^{\star}(X_d \setminus Z) = 0$. The set $G = \pi_d^{-1}(Z)$ is d-closed, d-separable and $|m|_d^{\star}(X \setminus G) = 0$, and so m is separable.

Conversely, let $m \in M_s(X)$ and let d be a bounded continuous ultra-pseudometric on X. Define $\mu = \mu_d : K(X_d) \to \mathbb{K}, \mu(A) = m(\pi_d^{-1}(A))$. Then μ is σ-additive.

Claim I: μ is τ-additive. Indeed, let (V_δ) be a net of clopen subsets of X_d with $V_\delta \downarrow \emptyset$. There exists a d-closed, d-separable subset G of X such that $|m|_d^{\star}(X \setminus G) = 0$. Given $\epsilon > 0$, there exists a sequence (Z_n) of d-clopen subsets with $X \setminus \subset \bigcup Z_n$ and $|m|(Z_n) < \epsilon$ for all n. We may assume that (Z_n) is decreasing. If $A_n = \pi_d(Z_n)$, then A_n is clopen in X_d and $|\mu|(A_n) \le \epsilon$. The set $M = \pi_d(G)$ is closed, separable and $X_d \setminus M \subset \bigcup A_n$. Thus $|\mu|^\star(X_d \setminus M) = 0$. Since M is separable, there exists an increasing sequence (δ_n) such that $M \subset \bigcup_{n=1}^\infty (X_d \setminus V_{\delta_n})$. Now $V_{\delta_n} \bigcap (X_d \setminus A_n) \downarrow \emptyset$. Since μ is σ-additive, there exists k such that $|\mu|(V_{\delta_k} \bigcap (X_d \setminus A_k)) < \epsilon$. Since $|\mu|(A_k) \le \epsilon$, it follows that $|\mu|(V_{\delta_k}) \le \epsilon$. This clearly proves that $\lim_\delta \mu(V_\delta) = 0$ and so μ is τ-additive.

Claim II: If d, d_1 are bounded continuous ultra-pseudometrics on X, with $d \le d_1$, and if $f \in C_b(X)$ is d-continuous, then for the functions

$$h : X_d \to \mathbb{K}, h_1 : X_{d_1} \to \mathbb{K}, h(\tilde{x}_d) = f(x) = h_1(\tilde{x}_{d_1}),$$

which are well defined and continuous, we have that $\int h d\mu_d = \int h_1 d\mu_{d_1}$. Indeed, let $\phi : X_{d_1} \to X_d, \tilde{x}_{d_1} \mapsto \tilde{x}_d$. Then ϕ is continuous and $\pi_d = \phi \circ \pi_{d_1}$. Let

$$S : (C_b(X_d, \beta) \to C_b(X_{d_1}, \beta), f \mapsto f \circ \phi.$$

Then S is continuous. Let $v_d : C_b(X_d) \to \mathbb{K}, v_d(g) = \int g d\mu_d$ and $v_{d_1} : C_b(X_{d_1}) \to \mathbb{K}, v_{d_1}(g) = \int g d\mu_{d_1}$. Then $S^\star v_{d_1} = v_d$. Since $\langle S^\star v_{d_1}, h\rangle =$

$\langle v_{d_1}, Sh\rangle = \langle v_{d_1}, h \circ \phi\rangle$, we have $\langle v_d, h\rangle = \langle v_{d_1}, h_1\rangle$, *i.e.* $\int h d\mu_d = \int h_1 d\mu_{d_1}$. This proves our claim.

For each $f \in C_b(X)$, there exists a bounded continuous ultra-pseudometric d on X such that f is d-continuous (e.g. $d(x,y) = |f(x) - f(y)|$). We define $u_m(f) = \int \tilde{f}_d d\mu_d$, where $\tilde{f}_d(\tilde{x}_d) = f(x)$. By claim II, u_m is well defined and linear. Also $u_m(T_d g) = \int g d\mu_d$ for each bounded continuous ultra-pseudometric d on X and each $g \in C_b(X_d)$.

Claim III: u_m is β-continuous on $C_b(X)$. Indeed, let $W = \{f \in C_b(X) : |u_m(f)| \leq 1\}$ and let d be a bounded continuous ultra-pseudometric on X. The set $V = \{g \in C_b(X_d) : |\int g d\mu_d| \leq 1\}$ is a β-neighborhood of zero in $C_b(X_d)$ and $T(V) \subset W$. Thus W is a β_e-neighborhood of zero, i.e. u_m is β_e-continuous.

Claim IV: If $f \in C_{rc}(X)$, then $u_m(f) = \int f dm$. Indeed, let $d(x,y) = |f(x) - f(y)|$ and let $h \in C_b(X_d), h(\tilde{x}_d) = f(x)$. Then, $u_m(f) = u_m(T_d h) = \int f d\mu_d$. It is easy to see that, for each clopen subset A of X, we have $u_m(\chi_A) = m(A)$. Since u_m is β_e-continuous, it is β_1-continuous. Thus there exists $m_1 \in M_\sigma(X)$ such that $u_m(f) = \int f dm_1$ for $f \in C_{rc}(X)$ Since $m_1(A) = u_m(\chi_A) = m(A)$ for each clopen set A, we have $m = m_1$ and so $u_m(f) = \int f dm$ for each $f \in C_{rc}(X)$. This completes the proof.

Theorem 6.5 *Let $m \in M_s(X)$ and let $(A_i)_{i\in I}$ be a clopen partition of X. Then: a) If B_i is a clopen subset of A_i and $B = \bigcup_i B_i$, then $m(B) = \sum_i m(B_i)$.*

b) For each $\epsilon > 0$, there exists a finite subset J of I such that $|m|\left(\bigcup_{i\notin J} A_i\right) \leq \epsilon$.

Proof: a) For each finite subset J of I, let $h_J = \sum_{i\in J} \chi_{B_i}, h = \sum_{i\in I} \chi_{B_i} = \chi_B$. The set $\{h_J : J \text{ finite subset of } I\} = H$ is uniformly bounded and equicontinuous. Hence $u_m(h_J) \to u_m(h)$, *i.e.* $\sum_{i\in J} m(B_i) \to m(B)$.

b) For each i, there exists a clopen subset B_i of A_i such that $|m(B_i)| \geq |m|(A_i)/2$. If $B = \bigcup B_i$, then $m(B) = \sum_i m(B_i)$. Given $\epsilon > 0$, there exists a finite subset J of I suh that $|m(B_i)| < \epsilon/2$ if $i \notin J$. So, for $i \notin J$, we have $|m|(A_i) \leq \epsilon$. If now A is a clopen subset of $D = \bigcup_{i\notin J} A_i$, then $A = \bigcup_{i\notin J} A_i \cap D$. By *a*), we have $m(A) = \sum_{i\notin J} m(A_i \cap D)$ and so $|m(A)| \leq \epsilon$. This proves that $|m|(D) \leq \epsilon$, which was to be shown.

Theorem 6.6 *If $m \in M_s(X)$, then every $f \in C_b(X)$ is m-integrable.*

Proof: We may assume that $\|m\| \leq 1$ and $\|f\| \leq 1$. Given $\epsilon > 0$, there exists a clopen partition $(A_i)_{i\in I}$ of X such that $|f(x) - f(y)| \leq \epsilon$ if x, y are in the same A_i. Let $J = \{i_1, \ldots, i_n\}$ be a finite subset of I such that $|m|(D) \leq \epsilon$, where $D = \bigcup_{i \notin J} A_i$. If now x, y are in some A_{i_k}, then $|f(x)-f(y)||m|(A_{i_k}) \leq \epsilon$. Also, if x, y are in D, then $|f(x)-f(y)||m|(D) \leq \epsilon$. The result now follows from Theorem 2.1.

Theorem 6.7 *If $f \in C_b(X)$ and $m \in M_s(X)$, then $u_m(f) = \int f dm$.*

Proof: We may assume that $\|m\| \leq 1$ and $\|f\| \leq 1$. Let $\epsilon > 0$. Since u_m is β_e-continuous and hence τ_u-continuous, there exists $0 < \epsilon_1 < \epsilon$ such that $|u_m(g)| < \epsilon$ if $\|g\| \leq \epsilon_1$. Let $(A_i)_{i\in I}$ be a clopen partition of X such that $|f(x) - f(y)| \leq \epsilon_1$ if x, y are in the same A_i. Let $x_i \in A_i, g = \sum_i f(x_i)\chi_{A_i}$. Then $\|f - g\| \leq \epsilon_1$ and so $|u_m(f - g)| < \epsilon$. For J finite subset of I, set $g_J = \sum_{i\in J} f(x_i)\chi_{A_i}$. The family $\{g_J : J \text{ finite subset of } I\}$ is uniformly bounded, equicontinuous and $g_J \to g$ pointwise. Thus $u_m(g_J) \to u_m(g)$. Using this and Theorem 6.5, we can choose a finite subset J of I such that $|u_m(g_J)-u_m(g)| \leq \epsilon$ and $|m|(D) \leq \epsilon$, where $D = \bigcup_{i\notin J} A_i$. Let $h = \sum_{i\notin J} f(x_i)\chi_{A_i}$. We have that $\left|\int f dm - \int g dm\right| \leq \|f - g\|\|m\| \leq \epsilon$ and $\left|\int g dm - \int g_J dm\right| = \left|\int h dm\right| \leq |m|(D) \leq \epsilon$. Since $u_m(g_J) = \int g_J dm$, it follows that $\left|\int f dm - u_m(f)\right| \leq \epsilon$, which proves that $\int f dm = u_m(f)$ since $\epsilon > 0$ was arbitrary.

Theorem 6.8 *Let τ be a locally convex topology on $C_b(X)$ which agrees with the topolohy τ_p of pointwise convergence on uniformly bounded equicontinuous subsets of $C_b(X)$. Then: (1) $C_{rc}(X)$ is τ-dense in $C_b(X)$.*
(2) τ is coarser than β_1.

Proof: (2) Assume that there exists a convex τ-neighborhood W of zero which is not a β_1-neighborhood. Then, there exists $h \in C_{rc}(X)$ such that W is not a β_Z-neighborhood of zero, where $Z = \{x \in \beta_o X : h^{\beta_o}(x) = 0\}$. Let $A_n = \{x \in X : |h(x)| \geq 1/n\}$. Since $\bar{A_n}^{\beta_o X}$ is disjoint from Z, there exist $r > 0$ and $f_n \in C_b(X)$ with $\|f_n\| \leq r, \|f_n\|_{A_n} \leq 1/n, f_n \notin W$. Let $x_o \in X$ and $\epsilon > 0$. Choose $n_o > 1/\epsilon$ with $x_o \in A_{n_o}$. If $n \geq n_o$ and $x \in A_{n_o}$, then $|f_n(x) - f_n(x_o)| \leq \epsilon$. Thus the sequence (f_n) is uniformly bounded and equicontinuous. Moreover, $f_n \to 0$ poitwise. By hypothesis, $f_n \to 0$ in the topology τ and so $f_n \in W$ eventually, a contradiction.
(1) Let $f \in C_b(X)$ and let W be a convex τ-neighborhood of zero. Since τ is coarser than the topology of uniform convergence, there exists $\epsilon > 0$ such

that

$$W_1 = \{g \in C_b(X) : \|g\| \leq \epsilon\} \subset W.$$

Let $(A_i)_{i \in I}$ be a clopen partition of X such that $|f(x) - f(y)| \leq \epsilon$ if x, y are in the same A_i. For each finite subset J of I, set $h_J = \sum_{i \in J} f(x_i)\chi_{A_i}$, where $x_i \in A_i$. If $h = \sum_{i \in I} f(x_i)\chi_{A_i}$, then the set $H = \{h_J : J \text{ finite}\}$ is uniformly bounded, equicontinuous and $h_J \to h$ poinstwise. Hence, there exists J finite such that $h - h_J \in W$. Since $f - h \in W_1 \subset W$, we get that $f - h_J \in W$, which clearly completes the proof.

Theorem 6.9 *For an $m \in M(X)$, the following are equivalent:(1) $m \in M_s(X)$.*
(2) For each clopen partition $(A_i)_{i \in I}$ of X, we have $m(X) = \sum_{i \in I} m(A_i)$.
(3) For each clopen partition $(A_i)_{i \in I}$ of X and each $\epsilon > 0$, there exists a finite subset J of I such that $|m| \left(\bigcup_{i \notin J} A_i\right) \leq \epsilon$.

Proof: (1) $\Rightarrow$ (2) For each finite subset J of I, set $h_J = \sum_{i \in J} \chi_{A_i}$. Then the family $\{h_J : J \text{ finite}\}$ is uniformly bounded and equicontinuous. Moreover $h_J \to 1$ poitwise. Hence $m(X) = \lim_J \int h_J dm = \sum_{i \in I} m(A_i)$.
(2) $\Rightarrow$ (3) For each i, there exists a clopen subset B_i of A_i such that $|m(B_i)| \geq |m|(A_i)/2$. If $B = \bigcup_i B_i$, then $m(X) = mq(X \setminus B) + \sum_i m(B_i$ and so $m(B) = \sum_i m(B_i)$ It follows easily from this that, given $\epsilon > 0$, there exists a finite subset J of I such that $|m|(A_i) \leq \epsilon$, for $i \notin J$, and $|m| \left(\bigcup_{i \notin J} A_i\right) \leq \epsilon$.
(3) $\Rightarrow$ (1) We show first that every $f \in C_b(X)$ is m-integrable. We may assume that $\|m\| \leq 1$ and $\|f\| \leq 1$. Given $\epsilon > 0$, there exists a clopen partition $(A_i)_{i \in I}$ of X such that $|f(x) - f(y)| \leq \epsilon$ if x, y are in the same A_i. Let J be a finite subset of I such that $|m|(D) \leq \epsilon$, where $D = \bigcup_{i \notin J} A_i$. If x, y are in the same A_i, then $|m|(A_i)|f(x) - f(y)| \leq \epsilon$. Also, if x, y are in D, then $|m|(D)|f(x) - f(y)| \leq \epsilon$. This, by Theorem 2.1, implies that f is m-integrable.We will finish the proof by showing that the map $\phi_m : C_b(X) \to \mathbb{K}, \phi_m(f) = \int f dm$, is β_u-cointinuous and hence β_e-cointinuous. In fact, let $(A_i)_{i \in I}$ be a clopen partition of X and $Q = \beta_o X \setminus \bigcup_{i \in I} \overline{A_i}^{\beta_o X}$. We need to show that the set $W = \{f \in C_b(X) : |\phi_m(f)| \leq 1\}$ is a β_Q-neighborhood of zero. So, let $r > 0$. There exists a finite subset J of I such that $|m|(D) < 1/r$, where $D = \bigcup_{i \notin J} A_i$. If $B = X \setminus D$, then $\bar{B}^{\beta_o X}$ is disjont from Q and $\{f \in C_b(X) : \|f\| \leq r, \|f\|_B \leq 1\} \subset W$. Hence the result follows.

Theorem 6.10 *For a linerar functional u on $C_b(X)$, the following are equiv-*

alent:
(1) u is β_e-continuous.
(2) u is β_u-continuous.
(3) u is τ_p-continuous on each uniformly bounded equicontinuous subset of $C_b(X)$.

Proof: It is clear that (1) implies (2). Also, since $\tau_p \leq \beta_u \leq \beta_e$ and since β_e coincides with τ_p on each uniformly bounded equicontinuous subset of $C_b(X)$, it follows that (2) implies (3). Finally, suppose that (3) holds. Let τ be the finest locally convex topology on $C_b(X)$ which agrees with τ_p on each uniformly bounded equicontinuous subset of $C_b(X)$. In view of Theorem 6.8, τ is coarser than β_1 and $C_{rc}(X)$ is τ-dense in $C_b(X)$. Let $m \in M_\sigma(X)$ be such that $u(f) = \int f dm$ for $f \in C_{rc}(X)$. We show that m is in $M_s(X)$. Indeed, let $(A_i)_{i\in I}$ be a clopen partition of X. For $J \subset I$ finite, set $h_J = \sum_{i\in J} \chi_{A_i}$. The family $\{h_J : J \text{ finite}\}\}$ is equicontinuous and $h_J \to 1$ in the topology τ_p. Thus $m(X) = u(1) = \lim_J u(h_J) = \sum_{i\in I} m(A_i)$. In view of the preceding Theorem, we get that $m \in M_s(X)$. Now u_m is β_e-continuous and hence τ-continuous. Also, $u = u_m$ on the τ-dense subset $C_{rc}(X)$ of $C_b(X)$. Thus $u = u_m$ on $C_b(X)$ and so u is β_e-continuous. This completes the proof.

Theorem 6.11 *An $m \in M(X)$ is separable iff, for each decreasing net (A_δ) of clopen subsets of X with $\bigcap \overline{A_\delta}^{\beta_o X} \in \Omega_u$, we have that $|m|(A_\delta|) \to 0$.*

Proof Assume that $m \in M_s(X)$. Then the linear map

$$\phi_m : C_b(X) \to \mathbb{K}, \phi_m(f) = \int f dm$$

is β_u-continuous. Let $Q = \bigcap \overline{A_\delta}^{\beta_o X} \in \Omega_u$, where (A_δ) is a decreasing net of clopen susets of X. Given $\epsilon > 0$, the set $V = \{f \in C_b(X) : |\phi_m(f)| \leq \epsilon\}$ is a β_u-neighborhood of zero. Hence, there exists $h \in C_Q$ such that $W = \{f \in C_b(X) : \|hf\| \leq 1\} \subset V$. The set $G = \{x \in \beta_o X : |h^{\beta_o}(x) \leq 1\}$ is clopen and contains Q. There exists δ_o such that $\overline{A_{\delta_o}}^{\beta_o X} \subset G$. Let D be a clopen subset of A_{δ_o}. Then $\chi_D \in W$ and so $|m(D)| \leq \epsilon$, which proves that $|m|(A_{\delta_o}) \leq \epsilon$. Conversely, suppose that the condition is satisfied and let $(A_i)_{i\in I}$ be a clopen partition of X. For $J \subset I$ finite, set $D_J = \bigcup_{i\in J} A_i, B_J = X \setminus D_J$. Then (B_J) is a decreasing net with $\bigcap_J \overline{B_J}^{\beta_o X} \in \Omega_u$. By our hypothesis, given $\epsilon > 0$, there exists a finite subset J of I such that $|m|(B_J) \leq \epsilon$. Now the result follows from Theorem 6.9.

Theorem 6.12 *Let $m \in M(X)$ and let $\bar{m} \in K(\beta_o X)$ be defined by $\bar{m}(Z) = m(Z \cap X)$. Then $m \in M_s(X)$ iff*

$$\inf\{|\bar{m}|(Z) : H \subset Z, Z \text{ clopen in } \beta_o X\} = 0$$

for each $H \in \Omega_u$.

Proof Assume that m is separable and let $H \in \Omega_u$. There exists a decreasing net (B_δ), of clopen subsets of $\beta_o X$, with $\bigcap B_\delta = H$. Let $A_\delta = B_\delta \cap X$. In view of the preceding Theorem, given $\epsilon > 0$, there exists δ such that $|\bar{m}|(B_\delta) = |m|(A_\delta) < \epsilon$. Conversely, suppose that the condition is satisfied and let (A_δ) be a decreasing net of clopen subsets of X with $\bigcap \overline{A_\delta}^{\beta_o X} = H \in \Omega_u$. By our hypothesis, given $\epsilon > 0$, there exists a clopen subset D of $\beta_o X$, containing H, such that $|\bar{m}|(D) < \epsilon$. Since $\bigcap \overline{A_\delta}^{\beta_o X} \subset D$, there exists δ such that $\overline{A_\delta}^{\beta_o X} \subset D$ and so $|m|(A_\delta) \leq |\bar{m}|(D) < \epsilon$. Hence the result follows from the preceding Theorem.

Theorem 6.13 *A subset H, of the dual space $M_s(X)$ of $(C_b(X), \beta_u)$, is β_u-equicontinuous iff $\sup_{m \in H} \|m\| < \infty$ and, for each decreasing net (A_δ) of clopen subsets of X with $\bigcap \overline{A_\delta}^{\beta_o X} \in \Omega_u$, we have that $|m|(A_\delta) \to 0$ uniformly for $m \in H$.*

Proof: Assume that H is β_u-equicontinuous. The polar H^o, of H in $C_b(X)$, is a β_u-neighborhood of zero and hence a τ_u-neighborhood. Thus, there exists $\epsilon > 0$ such that $\{f \in C_b(X) : \|f\| \leq \epsilon\} \subset H^o$. It follows from this that $\sup_{m \in H} \|m\| < \infty$. Let now (A_δ) be a net of clopen subsets of X with $\bigcap \overline{A_\delta}^{\beta_o X} = G \in \Omega_u$. If μ is a non-zero element of $\mathbb{K}$, then there exists $h \in C_G$ such that $\{f : \|hf\| \leq 1\} \subset \mu H^o$. As in the proof of Theorem 6.11, there exists δ such that $|m|(A_\delta) \leq |\mu|$ for all $m \in H$. Conversely, suppose that the condition is satisfied and let $Q \in \Omega_u$. There exists a decreasing net (A_δ) of clopen subsets of X such that $Q = \bigcap \overline{A_\delta}^{\beta_o X}$. Let $r > 0$ and choose $\alpha > \sup_m \|m\|$. By hypothesis, there exists δ such that $|m|(A_\delta) < 1/r$ for all $m \in H$. If $M = X \setminus A_{\delta_o}$, then $\bar{M}^{\beta_o X}$ is disjoint from Q and

$$\{f \in C_b(X) : \|f\| \leq r, \|f\|_M \leq 1/\alpha\} \subset H^o.$$

This proves that H^o is a β_Q-neighborhood of zero, for all $Q \in \Omega_u$, and hence it is a β_u-neighborhood of zero, which completes the proof.

Theorem 6.14 *Let τ be the finest locally convex topology on $C_b(X)$ which agrees with τ_p on uniformly bounded equicontinuous subsets of $C_b(X)$. For a subset H of $M_s(X)$, the following are equivalent: (1) H is β_u-equicontinuous.*
(2) H is β_e-equicontinuous.
(3) H is τ-equicontinuous

Proof: Since $\beta_u \leq \beta_e \leq \tau$, we only need to show that (3) implies (1). So, let H be τ-equicontinuous. Since $\tau \leq t_u$, we get that $\sup_{m\in H} \|m\| < \infty$. Let now $G \in \Omega_u$. There exists a clopen partition $(A_i)_{i\in I}$ of X such that G is disjoint from each $\overline{A_i}^{\beta_o X}$. For each $i \in I$, there exist $m_i \in H$ and a clopen subset B_i of A_i such that $|m_i(B_i)| \geq \sup_{m\in H} |m|(A_i)/2$. For each finite subset J of I, set $h_J = \sum_{i\in J} \chi_{A_i}, h = \chi_B$, where $B = \bigcup_i B_i$. Since the family $\{h_J : J \text{ finite}\}$ is uniformly bounded and equicontinuous and since $h_J \to h$ poinstwise, given a non-zero element μ of $\mathbb{K}$, there exists a finite subset J of I such that $h - h_j \in \mu H^o$. If now $i \notin J$, then $\chi_{B_i} \in \mu H^o$ and so $|m(B_i)| \leq |\mu|$, for all $m \in H$, which implies that $\sup_{m\in H} |m|(B_i) \leq 2|\mu|$. If $D = \bigcup_{i\notin J} A_i$, then for each clopen subset A of D and each $m \in H$ we have $m(A) = \sum_{i\notin J} m(A \cap A_i)$ and so $|m(A)| \leq 2|\mu|$, which implies that $|m|(D) \leq 2|\mu|$ for all $m \in H$. Let now $r > 0$. As it is shown above, there exists a finite subset J of I such that $\sup_{m\in H} |m|(D) \leq 1/r$, where $D = \bigcup_{i\notin J} A_i$. If $a > \sup_{m\in H} \|m\|$, and $B = \bigcup_{i\in J} A_i$, then $\bar{B}^{\beta_o X}$ is disjoint from G and

$$\{f \in C_b(X) : \|f\| \leq r, \|f\|_B \leq 1/a\} \subset H^o.$$

This proves that H^o is a β_G-neighborhood of zero for each $G \in \Omega_u$. The result now follows.

Recall that a $\mathbb{K}$-partition of unity for X is a family $(f_i)_{i\in I}$ of functions in $C_b(X)$ such that $\sum_i f_i(x) = 1$, for each $x \in X$, and the family $\{x \in X : f_i(x) \neq 0\}, i \in I$, is locally finite.

Definition 6.15 *A bounded linear functional ϕ on $C_b(X)$ is called u-additive if, for each uniformly bounded $\mathbb{K}$-partition of unity (f_i) for X, we have $\phi(1) = \sum_i \phi(f_i)$.*

Theorem 6.16 *A bounded linear functional ϕ on $C_b(X)$ is β_e-continuous iff it is u-additive .*

Proof: We may assume that $\|\phi\| \leq 1$. Define m on $K(X)$ by $m(A) = \phi(\chi_A)$. Then $m \in M(X)$ and $\|m\| \leq 1$. Assume now that ϕ is u-additive. Then, m

is separable. In fact, let (A_i) be a clopen partition of X. If $f_i = \chi_{A_i}$, then (f_i) is a $\mathbb{K}$-partition of unity for X. By our hypothersis, $m(X) = \phi(1) = \sum_i \phi(f_i) = \sum_i m(A_i)$ and so $m \in M_s(X)$ by Theorem 6.9. We need to show that $\phi(f) = \int f dm$ for each $f \in C_b(X)$. So let $f \in C_b(X)$. We may assume that $\|f\| \leq 1$. Given $\epsilon > 0$, there exists a clopen partition $(A_i)_{i \in I}$ of X such that $|f(x) - f(y)| \leq \epsilon$ if x, y are in the same A_i. Let $x_i \in A_i$, $f^* = \sum_{i \in I} f(x_i)\chi_{A_i}$. Since the family $\{f(x_i)\chi_{A_i} : i \in I\} \cup \{1 - f^*\}$ is a uniformly bounded $\mathbb{K}$-partition of unity for X, we get that $\phi(f^*) = \sum_i f(x_i)m(A_i)$. Given $\epsilon > 0$, there exists a finite subset $J = \{i_1, i_2, \ldots, i_n\}$ of I such that $|\phi(f^*) - \sum_{i \in J} f(x_i)m(A_i)| \leq \epsilon$ and $|m|(D) \leq \epsilon$, where $D = \bigcup_{i \notin J} A_i$. If x, y are in D, then $|f(x) - f(y)||m|(D) \leq \epsilon$. Similarly, if x, y are in some A_{i_k}, then $|f(x) - f(y)||m|(A_{i_k}) \leq \epsilon$. If $y \in D$, then (as in the proof of Theorem 2.1) we have

$$|\int f dm - \sum_{k=1}^{n} f(x_{i_k})m(A_{i_k}) - m(D)f(y)| \leq \epsilon$$

and so

$$|\int f dm - \sum_{k=1}^{n} f(x_{i_k})m(A_{i_k})| \leq \epsilon.$$

Since $|\phi(f - f^*)| \leq \|\phi\|\|f - f^*\| \leq \epsilon$, it follows that $|\phi(f) - \int f dm| \leq \epsilon$. This, being true for every $\epsilon > 0$, implies that $\phi(f) = \int f dm$.
Conversely, suppose that ϕ is β_e-continuous andf let $(f_i)_{i \in I}$ be a uniformly bounded $\mathbb{K}$-partition of unity for X. For J finite subset of I, set $h_J = \sum_{i \in J}$. The family $\{h_J : J \ \ finite\}$ is uniformly bounded and equicontinuous. Moreover $h_J \to 1$ poitwise. Since ϕ is β_e-continuous, we have that $\phi(1) = \lim \phi(h_J) = \sum_i \phi(f_i)$. This completes the proof.

Theorem 6.17 *The dual space $M_s(X)$ of $(C_b(X), \beta_e)$ is weak*-sequentially complete.*

Proof: Let (m_n) be a weak*-Cauchy sequence in $M_s(X)$ and let d be a bounded continuous ultra-pseudometric on X. For each n, there exists $\mu_n \in M_\tau(X_d)$ such that $\int g d\mu_n = \int T_d g dm_n$, for all $g \in C_b(X_d)$. In view of Theorem 3.6, there exists $\mu \in M_\tau(X_d)$ such that

$$\int g d\mu = \lim_{n \to \infty} \int g d\mu_n = \lim_{n \to \infty} \int T_d g dm_n,$$

for all $g \in C_b(X_d)$. If

$$\phi : C_b(X) \to \mathbb{K}, \phi(f) = \lim_{n\to\infty} \int f dm_n$$

and if $W = \{f \in C_b(X) : |\phi(f)| \leq 1\}$, then

$$\{g \in C_b(X_d) : \left|\int g d\mu\right| \leq 1\} \subset T_d^{-1}(W).$$

This proves that W is a β_e-neighborhood of zero and so ϕ is β_e-continuous. Thus, there exists $m \in M_s(X)$ such that

$$\int f dm = \phi(f) = \lim_{n\to\infty} \int f dm_n s$$

for all $f \in C_b(X)$. This completes the proof.

We will finish the paper with the following

Conjecture 6.18 *1)* $\beta_e = \beta_u$.
2) β_e *is the finest locally convex topology on* $C_b(X)$ *which coincides with the topology of simple convergence on uniformly bounded equicontinuous subsets of* $C_b(X)$.

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Singularities of $3-D$ Protter's problem for the wave equation involving lower order terms

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ABSTRACT. We investigate some boundary value problems (BVP) for the wave equation involving lower order terms which are three-dimensional analogues of Darboux-problems (or Cauchy-Goursat problems) on the plane. These problems, in the case of wave equation, have been formulated and studied by M. Protter (1954) in a $3-D$ domain Ω_0, bounded by two characteristic cones Σ_1 and $\Sigma_{2,0}$ and a plane region Σ_0. The homogeneous conditions are defind on Σ_1 and Σ_0. Many authors studied these problems using different methods, like: Wiener-Hopf method, special Legendre functions, a priori estimates, nonlocal regularization and others. It is well known that, for the infinite number of smooth functions in the right-hand side, these problems do not have classical solutions. Popivanov and Schneider (1995) discovered the reason of this fact for the case of Dirichlet's and Neumann's conditions on Σ_0: the strong power-type singularity appears in the generalized solution on the characteristic cone $\Sigma_{2,0}$. We investigated the case of third BVP on Σ_0 and obtained the existence of many singular solutions even for the wave equation involving lower order terms. Especially, for M. Protter's problems in $\mathbb{R}^3$ it is shown that for any $n \in N$ there exists a $C^n(\bar{\Omega}_0)$ - function, for which the corresponding unique generalized solution belongs to $C^n(\bar{\Omega}_0\backslash O)$ and has a strong power-type singularity at the point O. It is interesting that this singularity is isolated only at the vertex O of the characteristic cone $\Sigma_{2,0}$ and does not propagate along the cone. Furthermore, for the wave equation without lower order terms, we presented here the exact behavior of the singular solutions at the point O for the case of third BVP on Σ_0. Also, some weight a priori estimates are stated.

1. Introduction

Consider the hyperbolic partial differential equation involving the wave operator in its main part with lower order terms of the form

$$Lu \equiv u_{x_1x_1} + u_{x_2x_2} - u_{tt} + b_1u_{x_1} + b_2u_{x_2} + bu_t + cu = f \tag{1.1}$$

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expressed in Cartesian coordinates x_1, x_2, t in a simply connected region $\Omega_0 \subset \mathbb{R}^3$. The region

$$\Omega_0 := \{(x_1, x_2, t) : 0 < t < 1/2, t < \sqrt{x_1^2 + x_2^2} < 1 - t\}$$

is bounded by the disk

$$\Sigma_0 := \{(x_1, x_2, t) : t = 0, x_1^2 + x_2^2 < 1\}$$

and the characteristic surfaces of (1.1):

$$\Sigma_1 := \{(x_1, x_2, t) : 0 < t < 1/2, \sqrt{x_1^2 + x_2^2} = 1 - t\},$$
$$\Sigma_{2,0} := \{(x_1, x_2, t) : 0 < t < 1/2, \sqrt{x_1^2 + x_2^2} = t\}.$$

In this work we are interested in finding of sufficient conditions for the existence and uniqueness of a generalized solution of

Problem $\mathbf{P}_\alpha$. Find a solution of the equation (1.1) in Ω_0, which satisfies the boundary conditions

$$(1.2) \qquad P_\alpha : \quad u|_{\Sigma_1} = 0, \quad [u_t + \alpha u]|_{\Sigma_0} = 0,$$

where $\alpha \in C^1(\Sigma_0 \backslash (0,0,0))$.

The adjoint problem to $\mathbf{P}_\alpha$ is

Problem $\mathbf{P}^*_\alpha$. Find a solution of the adjoint equation

$$L^* u \equiv u_{x_1 x_1} + u_{x_2 x_2} - u_{tt} - (b_1 u)_{x_1} - (b_2 u)_{x_2} - (bu)_t + cu = g \text{ in } \Omega_0$$

with the boundary conditions:

$$(1.3) \qquad \mathbf{P}^*_\alpha : \quad u|_{\Sigma_{2,0}} = 0, \quad [u_t + (\alpha + b)u]|_{\Sigma_0} = 0 \quad .$$

The following problems, due to Protter [21], are known as

Protter's Problems. Find a solution of the wave equation

$$(1.4) \qquad \Box u \equiv \Delta_x u - u_{tt} \equiv u_{x_1 x_1} + u_{x_2 x_2} - u_{tt} = f \text{ in } \Omega_0$$

with the boundary conditions

$$(1.5) \qquad \begin{array}{llll} P1: & u|_{\Sigma_0 \cup \Sigma_1} = 0, & P1^*: & u|_{\Sigma_0 \cup \Sigma_{2,0}} = 0\,; \\ P2: & u|_{\Sigma_1} = 0, u_t|_{\Sigma_0} = 0, & P2^*: & u|_{\Sigma_{2,0}} = 0, u_t|_{\Sigma_0} = 0\,. \end{array}$$

The boundary conditions of problem $P1^*$ (respectively of $P2^*$) are the adjoint boundary conditions to such ones of $P1$ (respectively of $P2$) for the wave equation (1.4) in Ω_0. Protter [21] formulated and investigated problems $P1$ and $P1^*$ in Ω_0 as multi-dimensional analogues of the Darboux problem on the plane. It is well known that the corresponding Darboux problems in $\mathbb{R}^2$ are well posed, but this is not true for the Protter's problems in $\mathbb{R}^3$. For recent results concerning the problems (1.5) see papers of Popivanov, Schneider [20], Grammatikopoulos, Hristov, Popivanov [9] and references therein. For publications in this area see also: [1], [2], [7], [11], [14], [15], [16]. On the other hand, Bazarbekov [5] gives another analogue of the classical Darboux problem in the same domain Ω_0. Some different statements of Darboux type problems can be found in [4], [6], [13], [17] in bounded or unbounded domains different from Ω_0.

In case of Problem P_α with $\alpha(\varrho) \neq 0$ there are only few publications, while for Problem P_α, concerning the wave equation (1.4) see the results of [9]. Also, some results of this type can be found in Section 7 of the present paper.

In the case of the equation (1.1), which involves either lower order terms or some other type perturbations, Problem P_α in Ω_0 with $\alpha(\rho) \equiv 0$ has been studied by Aldashev in [**1**], [**2**], [**3**] and other his articles. For some comments, concerning Aldashev's results, we refer the reader to Remark 6.1 of Section 6.

Next, we formulate here the following well known result (see [**22**], [**19**]) presented in the terms of the polar coordinates $x_1 = \varrho\cos\varphi$, $x_2 = \varrho\sin\varphi, t$.

THEOREM 1.1. *For all $n \in \mathbb{N}, n \geq 4; a_n, b_n$ arbitrary constants, the functions*

$$v_n(\varrho, \varphi, t) = t\varrho^{-n}[\varrho^2 - t^2]^{n-\frac{3}{2}}(a_n \cos n\varphi + b_n \sin n\varphi) \tag{1.6}$$

are classical solutions of the homogeneous problem $P1^*$ and the functions

$$w_n(\varrho, \varphi, t) = \varrho^{-n}[\varrho^2 - t^2]^{n-\frac{1}{2}}(a_n \cos n\varphi + b_n \sin n\varphi) \tag{1.7}$$

are classical solutions of the homogeneous problem $P2^*$.

This theorem shows that for the classical solvability of the problem $P1$ (respectively, $P2$) the function f at least must be orthogonal to all smooth functions (1.6) (respectively, (1.7)). The reason of this fact has been found by Popivanov and Schneider in [**19**], where they announced for Problems $P1$ and $P2$ that there exist some singular solutions for the wave equation (1.4) with power type isolated singularity even for some very smooth functions f. Using Theorem 1.1, Popivanov and Schneider [**20**] proved the existence of some *generalized solutions* of Problems $P1$ and $P2$ which have at least power-type singularities at the vertex (0,0,0) of the cone $\Sigma_{2,0}$. For the homogeneous Problem P^*_α (except the case $\alpha \equiv 0$, i.e. except Problem $P2^*$) even for the wave equation we do not know nontrivial solutions analogous to (1.6) and (1.7). Anyway, in the present paper we prove some results which ensure the existence of many singular solutions. More precisely, in Theorems 6.1, 6.2 and 7.1 we give conditions for the coefficients under which we prove an analogous result for the general equation (1.1). Here we refer also to Khe Kan Cher [**15**], who gives some nontrivial solutions for the homogeneous Problems $P1^*$ and $P2^*$, but in the case of Euler-Poisson-Darboux equation. These are closely connected to the results of Theorem 1.1.

In order to obtain our results, we give the following definition of a solution of Problem P_α with a possible singularity at (0,0,0).

DEFINITION 1.1. *A function $u = u(x_1, x_2, t)$ is called a generalized solution of the problem*

$$P_\alpha: \qquad Lu = f, \qquad u\big|_{\Sigma_1} = 0, \qquad [u_t + \alpha(x)u]\big|_{\Sigma_0} = 0,$$

in Ω_0, if:

1) $u \in C^1(\bar{\Omega}_0 \setminus (0,0,0))$, $[u_t + \alpha(x)u]\big|_{\Sigma_0 \setminus (0,0,0)} = 0$, $u\big|_{\Sigma_1} = 0$,

2) the identity

$$\begin{aligned}&\int_{\Omega_0} [u_t v_t - u_{x_1} v_{x_1} - u_{x_2} v_{x_2} + (b_1 u_{x_1} + b_2 u_{x_2} + b u_t + cu - f)v] dx_1 dx_2 dt \\ &= \int_{\Sigma_0} \alpha(x)(uv)(x, 0) dx_1 dx_2,\end{aligned} \tag{1.8}$$

holds for all v in

$$V_0 := \{v \in C^1(\bar{\Omega}_0) : [v_t + (\alpha + b)v]\big|_{\Sigma_0} = 0,\ v = 0 \textit{ in a neighbourhood of } \Sigma_{2,0}\}.$$

In order to deal successfully with the encountered difficulties, as are the singularities on the cone $\Sigma_{2,0}$, we introduce the region

$$\Omega_\varepsilon = \Omega_0 \cap \{\varrho - t > \varepsilon\}, \quad \varepsilon \in [0,1),$$

which in polar coordinates becomes

$$\Omega_\varepsilon = \{(\varrho,\varphi,t) : t > 0,\ 0 \le \varphi < 2\pi,\ \varepsilon + t < \varrho < 1 - t\}. \tag{1.9}$$

and we define the notion of a *generalized solution* of Problem P_α in $\Omega_\varepsilon, \varepsilon \in (0,1)$ (see Definition 2.1). Note that, if a generalized solution u belongs to $C^1(\bar{\Omega}_\varepsilon) \cap C^2(\Omega_\varepsilon)$, it is called a *classical solution* of Problem P_α in Ω_ε, $\varepsilon \in (0,1)$, and it satisfies the wave equation (1.4) in Ω_ε. It should be pointed out that the case $\varepsilon = 0$ is totally different from the case $\varepsilon \neq 0$.

This paper is a generalization, extension and improvement of the results obtained in **[9]**. The paper, besides Introduction, consists of six more sections. In Section 2, using some appropriate techniques, we formulate the $2-D$ boundary value problems $P_{\alpha,1}$, $P_{\alpha,2}$ and $P_{\alpha,3}$, corresponding to the $3-D$ Problem P_α. The aim of Section 3 is to treat Problem $P_{\alpha,3}$. For this reason, we construct and study the system of integral equations assigned to the under consideration equation (1.1). Also we present results concerning the classical solutions of Problem $P_{\alpha,3}$ in $\Omega_\varepsilon, \varepsilon \in (0,1)$ and give corresponding a priori estimates. In Section 4 we prove Theorems 4.1 and 4.2 which ensure the existence and uniqueness of a generalized solution of Problem $P_{\alpha,1}$ in $2-D$ domain. Using the results of the previous section, in Section 5 we study the existence and uniqueness of a generalized solution of $3-D$ Problem P_α. More precisely, Theorem 5.1 ensures the uniqueness of a generalized solution of Problem P_α in $\Omega_\varepsilon, \varepsilon \in [0,1)$, while Theorems 5.2, 5.3 and 5.4 ensure the existence of a generalized solution for problem P_α in Ω_0, which is classical one in each domain $\Omega_\varepsilon, \varepsilon \in [0,1)$ and satisfies some a priori estimates in $C^2(\Omega_\varepsilon)$. Comparing these estimates with such ones of **[9]**, we see that the new estimates are better even for the case of wave equation without lower terms. In Theorems 6.1 and 6.2 of Section 6 under two different kind of conditions on the coefficients of the equation (1.1) we present some singular generalized solutions which are smooth enough away from the point $(0,0,0)$, while at the point $(0,0,0)$ they have power-type singularity of the type ϱ^{-n}. More precisely, in Theorem 6.1 we prove the following result:

Let $\alpha(\varrho) \ge 0$, $\alpha(\varrho) \in C^1([0,1])$; *the coefficients of the equation* (1.1) belong to $C^1(\bar{\Omega}_0)$ *and satisfy appropriate conditions. Then for each $n \in N$ there exists a function $f_n(\varrho,\varphi,t) \in C^{n-2}(\bar{\Omega}_0)$, for which the corresponding generalized solution u_n of the problem P_α belongs to $C^2(\bar{\Omega}_0 \backslash (0,0,0))$ and satisfies the estimate*

$$|u_n(\varrho,\varphi,\rho)| \ge \frac{1}{2}|u_n(2\varrho,\varphi,0)| + \varrho^{-n}|\cos n\varphi| \ge \varrho^{-n}|\cos n\varphi|,\ 0 < \varrho < 1. \tag{1.10}$$

In Theorem 6.2 under some other conditions on the coefficients of (1.1) and without any restrictions on the sign of $\alpha(\varrho)$, we arrive again to the same conclusion. Finally, in Theorem 7.1 of Section 7 we investigate the wave equation and find two-sided estimates for the singularities of generalized solution. In particular, the exact behavior for the singular solution $u_n(x_1,x_2,t)$ around (0,0,0) is $(x_1^2 + x_2^2)^{-n/2}$.

REMARK 1.1. *It is interesting that for any parameter $\alpha(x)$, involved in the boundary condition (1.2) on Σ_0 there are infinitely many singular solutions of the wave equation (Theorem 7.1). Note, that all these solutions have strong singularities at the vertex $(0,0,0)$ of the cone $\Sigma_{2,0}$. These singularities of generalized solutions do not propagate in the direction of the bicharacteristics on the characteristic cone. It is traditionally assumed that the wave equation with right-hand side sufficiently smooth in $\bar{\Omega}_0$ cannot have a solution with an isolated singular point. For results concerning the propagation of singularities for second order operators see Hörmander* [**10**], *Chapter 24.5. For some related results in the case of plane Darboux-Problem see* [**18**].

REMARK 1.2. *In 1960 Garabedian proved in* [**8**] *the uniqueness of classical solution of Problem P1. Existence and uniqueness results for a generalized solution of Problems P1 and P2 can be found in* [**20**].

REMARK 1.3. *Considering Problems P1 and P2, Popivanov and Schneider* [**19**] *announced the existence of singular solutions of both wave and degenerate hyperbolic equation. First a priori estimates for singular solutions of Protter's Problems P1 and P2, concerning the wave equation in $\mathbb{R}^3$, were obtained in* [**20**]. *In* [**1**] *Aldashev mentions the results of* [**19**] *and, for the case of the wave equation in $\mathbb{R}^{m+1}$, he shows that there exist solutions of Problem P1 (respectively, P2) in the domain Ω_ε, which grow up on the cones $\Sigma_{2,\varepsilon}$ like $\varepsilon^{-(n+m-2)}$ (respectively, $\varepsilon^{-(n+m-1)}$), when for $\varepsilon \to 0$ the cones $\Sigma_{2,\varepsilon} := \{\varrho = t + \varepsilon\}$ approximate $\Sigma_{2,0}$. It is obvious that for $m = 2$ this results can be compared with the estimate* (1.10) *of Theorem 6.1 and the analogous estimate of Theorem 6.2. Finally, we point out that in the case of an equation, which involves the wave operator and nonzero lower terms, Karatoprakliev* [**12**] *obtained a priori estimates, but only for the enough smooth solutions of Problem P1 in Ω_0.*

We conclude this section with the following

Open questions:

1. Find the exact behavior of all singular solutions at the point O, not only those ones which have been found in Theorems 6.1, 6.2 and 7.1.
2. Find some appropriate conditions for the function f under which the Problem P_α even for the wave equation has only regular solutions. We do not know any kind of such results even for the Problems $P1$ and $P2$.
3. In all results for the existence of singular solutions we assume that $a_2 \equiv 0$. Is it possible to find any singular solution when $a_2 \neq 0$? Even in the case $a_2 \neq 0$ Theorem 5.3 of Section 5 ensures the existence of a generalized solution for any function f, but we do not know the behavior of such solution at $(0,0,0)$.
4. From the a priori estimates obtained in Theorems 5.2-5.4, for all solutions of Problem P_α, including singular ones, it follows that none of these solutions can grow up faster than exponential one as $\rho \to 0$. The arising question is: are there singular solutions of Problem P_α with exponential growth as $\rho \to 0$ or any such solution is of polynomial growth satisfying (1.10)?

2. Preliminaries

In this section we consider the equation (1.1) in polar coordinates $x_1 = \varrho\cos\varphi$, $x_2 = \varrho\sin\varphi$, t

$$Lu = \frac{1}{\varrho}(\varrho u_\varrho)_\varrho + \frac{1}{\varrho^2}u_{\varphi\varphi} - u_{tt} + a_1 u_\varrho + a_2 u_\varphi + b u_t + cu = f \tag{2.1}$$

in a simply connected region

$$\Omega_\varepsilon := \{(\varrho,\varphi,t) : 0 < t < (1-\varepsilon)/2,\ 0 \le \varphi < 2\pi,\ \varepsilon + t < \varrho < 1 - t\}, \tag{2.2}$$

bounded by the disc $\Sigma_0 := \{(\varrho,\varphi,t) : t = 0, \varrho < 1\}$ and the characteristic surfaces of (2.1)

$$\Sigma_1 := \{(\varrho,\varphi,t) : 0 \le \varphi < 2\pi, \varrho = 1 - t\},$$
$$\Sigma_{2,\varepsilon} := \{(\varrho,\varphi,t) : 0 \le \varphi < 2\pi, \varrho = \varepsilon + t\}.$$

Obviously, the coefficients a_1, a_2 depend on b_1, b_2. We seek sufficient conditions for the existence and uniqueness of a generalized solution of the equation (2.1) with $f \in C(\bar\Omega_\varepsilon)$, which satisfies the following boundary conditions:

$$P_\alpha : \quad u\big|_{\Sigma_1\cap\partial\Omega_\varepsilon} = 0, \qquad [u_t + \alpha u]\big|_{\Sigma_0\cap\partial\Omega_\varepsilon} = 0; \tag{2.3}$$

$$P^*_\alpha : \quad u\big|_{\Sigma_{2,\varepsilon}} = 0, \qquad [u_t + (\alpha + b)u]\big|_{\Sigma_0\cap\partial\Omega_\varepsilon} = 0. \tag{2.4}$$

For the sake of simplicity, here we assume that all coefficients of (2.1) depend only on ϱ and t, and we set $\alpha(x) \equiv \alpha(|x|) = \alpha(\varrho) \in C^1((0,1])$. The problem P^*_α is the adjoint one to Problem P_α in Ω_ε.

REMARK 2.1. *In what follows, we consider the domain Ω_ε and its boundary in Cartesian coordinates. Nevertheless, for convenience we use the polar coordinates in the sense that the intersections $\varphi = 0$ and $\varphi = 2\pi$ do not belong to the boundary of Ω_ε and all the functions which we use here are considered as periodical ones.*

Now, in order to obtain our results, we define the notion of a generalized solution as follows.

DEFINITION 2.1. *A function $u = u(\varrho,\varphi,t)$ is called a generalized solution of Problem P_α in Ω_ε, $\varepsilon > 0$, if:*

1) $u \in C^1(\bar\Omega_\varepsilon)$, $u\big|_{\Sigma_1\cap\partial\Omega_\varepsilon} = 0$; $[u_t + \alpha(\varrho)u]\big|_{\Sigma_0\cap\partial\Omega_\varepsilon} = 0$;

2) the identity

$$\int_{\Omega_\varepsilon} [u_t v_t - u_\varrho v_\varrho - \frac{1}{\varrho^2}u_\varphi v_\varphi + (a_1 u_\rho + a_2 u_\varphi + b u_t + cu - f)v]\varrho\, d\varrho\, d\varphi\, dt \tag{2.5}$$
$$= \int_{\Sigma_0\cap\partial\Omega_\varepsilon} \alpha(\varrho)uv\varrho\, d\varrho d\varphi$$

holds for all

$$v \in V_\varepsilon := \{v \in C^1(\bar\Omega_\varepsilon) : [v_t + (\alpha + b)v]\big|_{\Sigma_0\cap\partial\Omega_\varepsilon} = 0, v\big|_{\Sigma_{2,\varepsilon}} = 0\}.$$

The following proposition describes the properties of generalized solutions of Problem P_α in Ω_ε.

LEMMA 2.1. *Each generalized solution of Problem P_α in Ω_0 is also a generalized solution of the same problem in Ω_ε for $\varepsilon > 0$.*

The proof of this lemma follows from the proof of Lemma 2.1 [9].

In the special, but main case, when

$$f(\varrho,\varphi,t)=f_n^{(1)}(\varrho,t)\cos n\varphi+f_n^{(2)}(\varrho,t)\sin n\varphi \tag{2.6}$$

we ask the generalized solution to be of the form

$$u(\varrho,\varphi,t)=u_n^{(1)}(\varrho,t)\cos n\varphi+u_n^{(2)}(\varrho,t)\sin n\varphi.$$

Then, in view of (2.1), we obtain the $2-D$ system

$$\begin{aligned}&\frac{1}{\varrho}(\varrho u^{(1)}_{n,\varrho})_\varrho-u^{(1)}_{n,tt}+a_1u^{(1)}_{n,\varrho}+bu^{(1)}_{n,t}+(c-\frac{n^2}{\varrho^2})u^{(1)}_n+na_2u^{(2)}_n=f^{(1)}_n,\\&\frac{1}{\varrho}(\varrho u^{(2)}_{n,\varrho})_\varrho-u^{(2)}_{n,tt}+a_1u^{(2)}_{n,\varrho}+bu^{(2)}_{n,t}+(c-\frac{n^2}{\varrho^2})u^{(2)}_n-na_2u^{(1)}_n=f^{(2)}_n.\end{aligned} \tag{2.7}$$

We consider this system in the domain

$$G_\varepsilon=\{(\varrho,t):t>0,\varepsilon+t<\varrho<1-t\}$$

which is bounded by the sets:

$$\begin{aligned}&S_0=\{(\varrho,t):t=0,0<\varrho<1\},\\&S_1=\{(\varrho,t):\varrho=1-t\},\quad S_{2,\varepsilon}=\{(\varrho,t):\varrho=t+\varepsilon\}.\end{aligned} \tag{2.8}$$

In this case, for $u=(u^{(1)},u^{(2)})(\varrho,t)$, the $2-D$ problem corresponding to P_α is

(2.9)

$$P_{\alpha,1}:\begin{cases}\frac{1}{\varrho}(\varrho u^{(1)}_{\varrho})_\varrho-u^{(1)}_{tt}+a_1u^{(1)}_{\varrho}+bu^{(1)}_{t}+(c-\frac{n^2}{\varrho^2})u^{(1)}+na_2u^{(2)}=f^{(1)}\text{ in }G_\varepsilon,\\ \frac{1}{\varrho}(\varrho u^{(2)}_{\varrho})_\varrho-u^{(2)}_{tt}+a_1u^{(2)}_{\varrho}+bu^{(2)}_{t}+(c-\frac{n^2}{\varrho^2})u^{(2)}-na_2u^{(1)}=f^{(2)}\text{ in }G_\varepsilon,\\ u^{(i)}|_{S_1\cap\partial G_\varepsilon}=0,\qquad [u^{(i)}_t+\alpha(\varrho)u^{(i)}]|_{S_0\cap\partial G_\varepsilon}=0,\quad i=1,2.\end{cases}$$

The generalized solution of the Problem $P_{\alpha,1}$ is defined by the

DEFINITION 2.2. *A function $u=(u^{(1)},u^{(2)})(\varrho,t)$ is called a generalized solution of Problem $P_{\alpha,1}$ in G_ε, $\varepsilon>0$, if:*

1) $u\in C^1(\bar G_\varepsilon)$, $[u^{(i)}_t+\alpha(\varrho)u^{(i)}]|_{S_0\cap\partial G_\varepsilon}=0$, $u^{(i)}|_{S_1\cap\partial G_\varepsilon}=0, i=1,2$;

2) the identities

$$\begin{aligned}&\int_{G_\varepsilon}\Big[u^{(1)}_tv_{1,t}-u^{(1)}_\varrho v_{1,\varrho}+\Big(a_1u^{(1)}_\varrho+bu^{(1)}_t+(c-\frac{n^2}{\varrho^2})u^{(1)}+na_2u^{(2)}\\&\quad-f^{(1)}\Big)v_1\Big]\,\varrho d\varrho\,dt=\int_{S_0\cap\partial G_\varepsilon}\alpha(\varrho)u^{(1)}v_1\varrho\,d\varrho,\\&\int_{G_\varepsilon}\Big[u^{(2)}_tv_{2,t}-u^{(2)}_\varrho v_{2,\varrho}+\Big(a_1u^{(2)}_\rho+bu^{(2)}_t+(c-\frac{n^2}{\varrho^2})u^{(2)}-na_2u^{(1)}\\&\quad-f^{(2)}\Big)v_2\Big]\,\varrho d\varrho\,dt=\int_{S_0\cap\partial G_\varepsilon}\alpha(\varrho)u^{(2)}v_2\varrho\,d\varrho\end{aligned} \tag{2.10}$$

hold for all

$$v_1,v_2\in V^{(1)}_\varepsilon=\{v\in C^1(\bar G_\varepsilon):[v_t+(\alpha+b)v]|_{S_0\cap\partial G_\varepsilon}=0,v|_{S_{2,\varepsilon}}=0\}.$$

Introducing a new function

$$z^{(i)}(\varrho,t)=\varrho^{\frac{1}{2}}u^{(i)}(\varrho,t),\ i=1,2, \tag{2.11}$$

we transform the system (2.9) to the system

$$\begin{aligned} z^{(1)}_{\varrho\varrho}-z^{(1)}_{tt}+a_1z^{(1)}_{\varrho}+bz^{(1)}_{t}+\left(c-\frac{1}{2\varrho}a_1-\frac{4n^2-1}{4\varrho^2}\right)z^{(1)}+na_2z^{(2)}=\varrho^{\frac{1}{2}}f^{(1)},\\ z^{(2)}_{\varrho\varrho}-z^{(2)}_{tt}+a_1z^{(2)}_{\varrho}+bz^{(2)}_{t}+\left(c-\frac{1}{2\varrho}a_1-\frac{4n^2-1}{4\varrho^2}\right)z^{(2)}-na_2z^{(1)}=\varrho^{\frac{1}{2}}f^{(2)}. \end{aligned} \tag{2.12}$$

with the string operator in the main part. Substituting the new coordinates

$$\xi=1-\varrho-t,\eta=1-\varrho+t, \tag{2.13}$$

from (2.12) we derive

$$\begin{cases} U^{(1)}_{\xi\eta}-A_1U^{(1)}_{\xi}-B_1U^{(1)}_{\eta}-C_1U^{(1)}-D_1U^{(2)}=F^1(\xi,\eta) \text{ in } D_\varepsilon,\\ U^{(2)}_{\xi\eta}-A_2U^{(2)}_{\xi}-B_2U^{(2)}_{\eta}-C_2U^{(2)}-D_2U^{(1)}=F^2(\xi,\eta) \text{ in } D_\varepsilon, \end{cases} \tag{2.14}$$

where $D_\varepsilon=\{(\xi,\eta):0<\xi<\eta<1-\varepsilon\}$ and for $i=1,2$

$$\begin{aligned} U^{(i)}(\xi,\eta)&=z^{(i)}(\varrho(\xi,\eta),t(\xi,\eta)),\\ F^{(i)}(\xi,\eta)&=\tfrac{1}{4\sqrt{2}}(2-\eta-\xi)^{\frac{1}{2}}f^{(i)}(\varrho(\xi,\eta),t(\xi,\eta)). \end{aligned} \tag{2.15}$$

Note that, in the case of the equation (2.1), the corresponding coefficients of the system (2.14) are:

$$\begin{aligned} &A_1=A_2=\frac{1}{4}(a_1+b),B_1=B_2=\frac{1}{4}(a_1-b),D_2=-D_1=\frac{1}{4}na_2,\\ &C_1=C_2=\frac{1}{4}\left(\frac{4n^2-1}{(2-\xi-\eta)^2}+\frac{1}{2-\xi-\eta}a_1-c\right). \end{aligned} \tag{2.16}$$

As we see, the Problem $P_{\alpha,1}$ is reduced to the Darboux-Goursat problem for the system (2.14) with the same boundary conditions. That is, we become the

Problem $P_{\alpha,2}$. Solve in D_ε the system (2.14) with the boundary conditions

$$U^{(i)}(0,\eta)=0,(U^{(i)}_\eta-U^{(i)}_\xi)(\xi,\xi)+\alpha(1-\xi)U^{(i)}(\xi,\xi)=0,\quad i=1,2. \tag{2.17}$$

In order to investigate the smoothness or the singularity of a solution of the original $3-D$ problem P_α on $\Sigma_{2,0}$, we are seeking for a classical solution of the corresponding $2-D$ problem $P_{\alpha,2}$ not only in the domain D_ε, but also in the domain

$$D^{(1)}_\varepsilon:=\{(\xi,\eta):0<\xi<\eta<1,0<\xi<1-\varepsilon\},\quad \varepsilon>0. \tag{2.18}$$

Clearly, $D_\varepsilon\subset D^{(1)}_\varepsilon$.

Consider now an appropriate boundary value problem for the system of equations (2.14), i.e.

$$P_{\alpha,3}:\quad \begin{cases} U^{(1)}_{\xi\eta}-A_1U^{(1)}_{\xi}-B_1U^{(1)}_{\eta}-C_1U^{(1)}-D_1U^{(2)}=F^1(\xi,\eta) \text{ in } D^{(1)}_\varepsilon,\\ U^{(2)}_{\xi\eta}-A_2U^{(2)}_{\xi}-B_2U^{(2)}_{\eta}-C_2U^{(2)}-D_2U^{(1)}=F^2(\xi,\eta) \text{ in } D^{(1)}_\varepsilon,\\ \\ U^{(i)}(0,\eta)=0,(U^{(i)}_\eta-U^{(i)}_\xi)(\xi,\xi)+\alpha(1-\xi)U^{(i)}(\xi,\xi)=0,\ i=1,2, \end{cases} \tag{2.19}$$

whose coefficients are continuous in $\bar{D}_\varepsilon^{(1)}$, $\varepsilon > 0$.

3. The system of integral equations corresponding to Problem $P_{\alpha,3}$

First of all we construct an equivalent to the problem $P_{\alpha,3}$ system of integral equations in such a way that any solution of the problem $P_{\alpha,3}$ to be also a solution of the constructed system. For this reason, for any $(\xi_0, \eta_0) \in D_\varepsilon^{(1)}$, we consider the sets

$$\Pi := \{(\xi,\eta) : 0 < \xi < \xi_0, \xi_0 < \eta < \eta_0\}, \quad T := \{(\xi,\eta) : 0 < \xi < \eta, 0 < \eta < \xi_0\}$$

and the following integrals:

$$I_0^{(i)} := \iint_\Pi U_{\xi\eta}^{(i)}(\xi,\eta)\, d\eta\, d\xi = \int_0^{\xi_0} \int_{\xi_0}^{\eta_0} U_{\xi\eta}^{(i)}(\xi,\eta)\, d\eta\, d\xi$$

and

$$I_1^{(i)} := \iint_T U_{\xi\eta}^{(i)}(\xi,\eta)\, d\eta\, d\xi = \int_0^{\xi_0} \int_{\xi}^{\xi_0} U_{\xi\eta}^{(i)}(\xi,\eta)\, d\eta\, d\xi.$$

As it has been shown in [**9**],

$$I_0^{(i)} + 2I_1^{(i)} = U^{(i)}(\xi_0,\eta_0) - \int_0^{\xi_0} \alpha(1-\xi)U^{(i)}(\xi,\xi)\, d\xi.$$

Set $p^{(i)} := U_\xi^{(i)}$, $q^{(i)} := U_\eta^{(i)}$. Then, in view of the last relation and (2.19), we obtain

$$\begin{aligned} U^{(1)}(\xi_0,\eta_0) = & \int_0^{\xi_0} \int_{\xi_0}^{\eta_0} [F^1 + A_1 p^{(1)} + B_1 q^{(1)} + C_1 U^{(1)} + D_1 U^{(2)}](\xi,\eta)\, d\eta\, d\xi \\ & +2 \int_0^{\xi_0} \int_0^{\eta} [F^1 + A_1 p^{(1)} + B_1 q^{(1)} + C_1 U^{(1)} + D_1 U^{(2)}](\xi,\eta)\, d\xi\, d\eta \\ & + \int_0^{\xi_0} \alpha(1-\xi)U^{(1)}(\xi,\xi)\, d\xi, \quad \text{for } (\xi_0,\eta_0) \in \bar{D}_\varepsilon^{(1)}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} U^{(2)}(\xi_0,\eta_0) = & \int_0^{\xi_0} \int_{\xi_0}^{\eta_0} [F^2 + A_2 p^{(2)} + B_2 q^{(2)} + C_2 U^{(2)} + D_2 U^{(1)}](\xi,\eta)\, d\eta\, d\xi \\ & +2 \int_0^{\xi_0} \int_0^{\eta} [F^2 + A_2 p^{(2)} + B_2 q^{(2)} + C_2 U^{(2)} + D_2 U^{(1)}](\xi,\eta)\, d\xi\, d\eta \\ & + \int_0^{\xi_0} \alpha(1-\xi)U^{(2)}(\xi,\xi)\, d\xi, \quad \text{for } (\xi_0,\eta_0) \in \bar{D}_\varepsilon^{(1)}, \end{aligned} \tag{3.2}$$

which is the first couple of desired integral equations. From (3.1) and (3.2) we derive for the first derivatives $p^{(i)}$ and $q^{(i)}$ $(i = 1, 2)$ the next four integral equations:

$$\begin{aligned} p^{(1)}(\xi_0,\eta_0) &= \int_0^{\xi_0} [F^1 + A_1 p^{(1)} + B_1 q^{(1)} + C_1 U^{(1)} + D_1 U^{(2)}](\xi,\xi_0)\,d\xi \\ &+ \int_{\xi_0}^{\eta_0} [F^1 + A_1 p^{(1)} + B_1 q^{(1)} + C_1 U^{(1)} + D_1 U^{(2)}](\xi_0,\eta)\,d\eta \\ &+\alpha(1-\xi_0)U^{(1)}(\xi_0,\xi_0), \end{aligned} \tag{3.3}$$

$$q^{(1)}(\xi_0,\eta_0) = \int_0^{\xi_0} [F^1 + A_1 p^{(1)} + B_1 q^{(1)} + C_1 U^{(1)} + D_1 U^{(2)}](\xi,\eta_0)\,d\xi, \tag{3.4}$$

$$\begin{aligned} p^{(2)}(\xi_0,\eta_0) &= \int_0^{\xi_0} [F^2 + A_2 p^{(2)} + B_2 q^{(2)} + C_2 U^{(2)} + D_2 U^{(1)}](\xi,\xi_0)\,d\xi \\ &+ \int_{\xi_0}^{\eta_0} [F^2 + A_2 p^{(2)} + B_2 q^{(2)} + C_2 U^{(2)} + D_2 U^{(1)}](\xi_0,\eta)\,d\eta \\ &+\alpha(1-\xi_0)U^{(2)}(\xi_0,\xi_0), \end{aligned} \tag{3.5}$$

$$q^{(2)}(\xi_0,\eta_0) = \int_0^{\xi_0} [F^2 + A_2 p^{(2)} + B_2 q^{(2)} + C_2 U^{(2)} + D_2 U^{(1)}](\xi,\eta_0)\,d\xi. \tag{3.6}$$

Set now

$$\begin{aligned} M &:= \max\left(\sup_{D_\varepsilon^{(1)}}|F^1|, \sup_{D_\varepsilon^{(1)}}|F^2|\right), \quad M_\alpha := \sup_{[0,1-\varepsilon]}|\alpha(\xi)| \\ c(\varepsilon) &:= \max_{i=1,2}\left\{\sup_{D_\varepsilon^{(1)}}|A_i|,\ \sup_{D_\varepsilon^{(1)}}|B_i|,\ \sup_{D_\varepsilon^{(1)}}|C_i|,\ \sup_{D_\varepsilon^{(1)}}|D_i|\right\}, \end{aligned} \tag{3.7}$$

and state the following results

THEOREM 3.1. *Let $F^i, A_i, B_i, C_i, D_i \in C(\bar{D}_\varepsilon^{(1)})$, $i = 1,2$, $\varepsilon > 0$. Then there exists a classical solution $(U^{(1)}, U^{(2)}) \in C^1(\bar{D}_\varepsilon^{(1)})$ of the Problem $P_{\alpha,3}$ for which $U_{\xi\eta}^{(i)}(\xi,\eta) \in C(\bar{D}_\varepsilon^{(1)}), i = 1,2$ and*

$$\begin{aligned} &|U^{(i)}(\xi_0,\eta_0)| \le M[4c(\varepsilon) + M_\alpha]^{-2}\exp[8c(\varepsilon) + 2M_\alpha] \quad \text{in } D_\varepsilon^{(1)},\ i = 1,2, \\ &\sup_{D_\varepsilon^{(1)}}\{|U_\xi^{(i)}|, |U_\eta^{(i)}|\} \le M[4c(\varepsilon) + M_\alpha]^{-1}\exp[8c(\varepsilon) + 2M_\alpha],\ i = 1,2. \end{aligned} \tag{3.8}$$

Proof. In order to get our results, we will solve the system of integral equations (3.1) – (3.6). For this reason we use sequence of successive approximations $U^{(n)}$, defined by the formula

$$U_{m+1}^{(i)}(\xi_0,\eta_0) = \int_0^{\xi_0}\int_{\xi_0}^{\eta_0} E_m^{(i)}(\xi,\eta)\,d\eta\,d\xi + 2\int_0^{\xi_0}\int_0^{\eta} E_m^{(i)}(\xi,\eta)\,d\xi\,d\eta$$
$$+\int_0^{\xi_0} \alpha(1-\xi)U_m^{(i)}(\xi,\xi)\,d\xi,\ i=1,2$$
$$p_{m+1}^{(i)}(\xi_0,\eta_0) = \int_0^{\xi_0} E_m^{(i)}(\xi,\xi_0)\,d\xi + \int_{\xi_0}^{\eta_0} E_m^{(i)}(\xi_0,\eta)\,d\eta \tag{3.9}$$
$$+\alpha(1-\xi_0)U_m^{(i)}(\xi_0,\xi_0),\ i=1,2,$$
$$q_{m+1}^{(i)}(\xi_0,\eta_0) = \int_0^{\xi_0} E_m^{(i)}(\xi,\eta_0)\,d\xi,\quad i=1,2,$$

$$U_0^{(i)}(\xi_0,\eta_0)=0,\ p_0^{(i)}(\xi_0,\eta_0)=0,\ q_0^{(i)}(\xi_0,\eta_0)=0,\quad i=1,2,\quad \text{in } D_\varepsilon^1,$$

where

$$E_m^{(1)}(\xi,\eta)\ :\ = [F^1 + A_1p_m^{(1)} + B_1q_m^{(1)} + C_1U_m^{(1)} + D_1U_m^{(2)}](\xi,\eta),$$
$$E^{(2)}(\xi,\eta)\ :\ = [F^2 + A_2p_m^{(2)} + B_2q_m^{(2)} + C_2U_m^{(2)} + D_2U_m^{(1)}](\xi,\eta).$$

We will show that each one of the functions $U^{(i)}, p^{(i)}$ and $q^{(i)}, i=1,2$, is continuous in $\bar{D}_\varepsilon^{(1)}$ and for any $(\xi_0,\eta_0)\in\bar{D}_\varepsilon^{(1)}$ and $m\in\mathbb{N}$

$$|(U_m^{(i)} - U_{m-1}^{(i)})(\xi_0,\eta_0)| \le M\frac{[4c(\varepsilon)+M_\alpha]^{m-1}}{(m+1)!}\,(\xi_0+\eta_0)^{m+1}, \tag{3.10}$$

$$\left\{|(p_m^{(i)} - p_{m-1}^{(i)})|(\xi_0,\eta_0),|(q_m^{(i)} - q_{m-1}^{(i)})|(\xi_0,\eta_0)\right\}|$$
$$\le M\frac{[4c(\varepsilon)+M_\alpha]^{m-1}}{m!}\,(\xi_0+\eta_0)^m \tag{3.11}$$

Indeed, by induction: 1) For $m=1$

$$U_1^{(i)}(\xi_0,\eta_0) = \int_0^{\xi_0}\int_{\xi}^{\eta_0} F^{(i)}(\xi,\eta)\,d\eta\,d\xi + \int_0^{\xi_0}\int_{\xi}^{\xi_0} F^{(i)}(\xi,\eta)\,d\eta\ d\xi\,,$$

and hence

$$|U^{(1)}(\xi_0,\eta_0)| \le M\xi_0\eta_0 \le M(\xi_0+\eta_0)^2/2.$$

Similarly can be estimated $p_1^{(i)}$ and $q_1^{(i)}$.

2) Let now, by the induction hypothesis, (3.10) and (3.11) be satisfied for some $m\in\mathbb{N}$. Then for $i=1,2$

$$|(U_m^{(i)} - U_{m-1}^{(i)})(\xi_0,\eta_0)| \le M\frac{[4c(\varepsilon)+M_\alpha]^{m-1}}{m!}(\xi_0+\eta_0)^m := Q_m(\xi_0+\eta_0)^m.$$

It follows that

$$|(U_{m+1}^{(i)} - U_m^{(i)})(\xi_0,\eta_0)| \le Q_m\left[4c(\varepsilon)\left(\int_0^{\xi_0}\int_{\xi_0}^{\eta_0}(\xi+\eta)^m\,d\eta\,d\xi\right.\right.$$
$$\left.\left.+2\int_0^{\xi_0}\int_0^{\eta}(\xi+\eta)^m\,d\xi\,d\eta\right) + \frac{M_\alpha}{m+1_\alpha}\int_0^{\xi_0}(2\xi)^{m+1}\,d\xi\right]$$

$$\begin{aligned}
&\le \frac{Q_m}{(m+1)(m+2)}\left[4c(\varepsilon)\Big((\xi_0+\eta_0)^{m+2}-\eta_0^{m+2}-\xi_0^{m+2}\Big)+M_\alpha(2\xi_0)^{m+2}\right]\\
&\le \frac{Q_{m+1}}{(m+2)}(\xi_0+\eta_0)^{m+2}.
\end{aligned}$$

By (3.3) and (3.5), we have

$$|(p^{(i)}_{m+1}-p^{(i)}_m)|(\xi_0,\eta_0)$$

$$\le \frac{Q_m}{m+1}\left[4c(\varepsilon)\Big((\xi_0+\eta_0)^{m+1}-\xi_0{}^{m+1}\Big)+M_\alpha(2\xi_0)^{m+1}\right]\le Q_{m+1}(\xi_0+\eta_0)^{m+1}.$$

Similar estimate follows for $(q^{(i)}_{m+1}-q^{(i)}_m)$. So, the inequalities (3.10) and (3.11) hold and hence the uniform convergence of the sequences $\{U^{(i)}_m(\xi,\eta)\}_{m\in\mathbb{N}}$, $\{p^{(i)}_m(\xi,\eta)\}_{m\in\mathbb{N}}$ and $\{q^{(i)}_m(\xi,\eta)\}_{m\in\mathbb{N}}$ in $\bar{D}^{(1)}_\varepsilon$ is obvious. For the limit functions $U^{(i)},p^{(i)},q^{(i)}\in C(\bar{D}^{(1)}_\varepsilon)$ we obtain the integral equalities (3.1) – (3.6) with obvious condition $U^{(i)}(0,\eta_0)=0$. From the integral equalities (3.1) – (3.6), it follows that $p^{(i)}=U^{(i)}_\xi$ and $q^{(i)}=U^{(i)}_\eta$ in $\bar{D}^{(1)}_\varepsilon$. Therefore, $U^{(i)}\in C^1(\bar{D}^{(1)}_\varepsilon), i=1,2$.

Also, in view of (3.10), we see that

$$\begin{aligned}
|(U^{(i)}(\xi_0,\eta_0)| &= \left|\sum_{m=0}^{\infty}(U^{(i)}_{m+1}-U^{(i)}_m)(\xi_0,\eta_0)\right| \le M\sum_{m=0}^{n}\frac{[4c(\varepsilon)+M_\alpha]^m}{(m+2)!}(\xi_0+\eta_0)^{m+2}\\
&\le M[4c(\varepsilon)+M_\alpha]^{-2}\exp[8c(\varepsilon)+2M_\alpha],\quad i=1,2.
\end{aligned}$$

So, using (3.11), for the derivatives $U^{(i)}_{\xi_0}(\xi_0,\eta_0)$ and $U^{(i)}_{\eta_0}(\xi_0,\eta_0)$we get the estimates:

$$\begin{aligned}
|U^{(i)}_{\xi_0}(\xi_0,\eta_0)| &= |\sum_{m=0}^{\infty}(p^{(i)}_{m+1}-p^{(i)}_m)(\xi_0,\eta_0)|\le M\sum_{m=0}^{\infty}\frac{[4c(\varepsilon)+M_\alpha]^m}{(m+1)!}(\xi_0+\eta_0)^{m+1}\\
&\le M[4c(\varepsilon)+M_\alpha]^{-1}\exp[8c(\varepsilon)+2M_\alpha],\quad i=1,2
\end{aligned}$$

and

$$|U^{(i)}_{\eta_0}(\xi_0,\eta_0)|\le M[4c(\varepsilon)+M_\alpha]^{-1}\exp[8c(\varepsilon)+2M_\alpha],\quad i=1,2,$$

which shows (3.8). Also, by (3.3) – (3.6), it follows that

$$\begin{aligned}
U^{(1)}_{\xi_0\eta_0} &\equiv U^{(1)}_{\eta_0\xi_0}=F^1+A_1U^{(1)}_\xi+B_1U^{(1)}_\eta+C_1U^{(1)}+D_1U^{(2)},\\
U^{(2)}_{\xi_0\eta_0} &\equiv U^{(2)}_{\eta_0\xi_0}=F^2+A_2U^{(2)}_\xi+B_2U^{(2)}_\eta+C_2U^{(2)}+D_2U^{(1)}.
\end{aligned}$$

Thus, the vector-function $U(\xi_0,\eta_0)$ is a solution of the system from (2.19) and $U_{\xi\eta}\in C(\bar{D}^{(1)}_\varepsilon)$. Finally, using representations (3.3) – (3.6) of the first derivatives of $U^{(i)}$, we conclude that each function $U^{(i)}(\xi_0,\eta_0)$ satisfies the boundary condition (2.19) of the Problem $P_{\alpha,3}$ for $\eta=\xi$. ∎

The next lemma is very important for the investigation of the singularity of a generalized solution of problem P_α.

LEMMA 3.1. *Let* $F^i, A_i, B_i, C_i, D_i \in C(\bar{D}_\varepsilon^{(1)})$, $i = 1, 2$,

$$A_i \geq 0, B_i \geq 0, C_i \geq 0, D_i \geq 0, \ \alpha(1-\xi) \geq 0 \ \textit{in } \bar{D}_\varepsilon^{(1)}, \ \ i = 1,2 \tag{3.12}$$

and

$$\text{(a) } p_1^{(i)} \geq 0 \ \textit{ and } \ q_1^{(i)} \geq 0, \qquad \textit{or} \ \ \text{(b) } F^{(i)} \geq 0 \ \textit{in } \bar{D}_\varepsilon^{(1)}, \quad i = 1, 2. \tag{3.13}$$

Then for the solution $(U^{(1)}, U^{(2)})$ *of Problem* $P_{\alpha,3}$ *(already found in Theorem 3.1) we have*

$$U^{(i)}(\xi,\eta) \geq 0, \quad U_\eta^{(i)}(\xi,\eta) \geq 0, \quad U_\xi^{(i)}(\xi,\eta) \geq 0 \quad \textit{for } (\xi,\eta) \in \bar{D}_\varepsilon^{(1)}, \ i = 1, 2. \tag{3.14}$$

Proof. Suppose first that the more stronger condition (**b**) is satisfied. Then, in view of (3.9), for $(\xi_0, \eta_0) \in \bar{D}_\varepsilon^{(1)}$ we have

$$U_1^{(i)}(\xi_0,\eta_0) = \int_0^{\xi_0} \int_{\xi_0}^{\eta_0} F^{(i)}(\xi,\eta)\, d\eta\, d\xi + 2\int_0^{\xi_0} \int_0^{\eta} F^{(i)}(\xi,\eta)\, d\xi\, d\eta \geq 0, \tag{3.15}$$

$$p_1^{(i)}(\xi_0,\eta_0) = \int_0^{\xi_0} F^{(i)}(\xi,\xi_0)\, d\xi + \int_{\xi_0}^{\eta_0} F^{(i)}(\xi_0,\eta)\, d\eta \geq 0, \tag{3.16}$$

$$q_1^{(i)}(\xi_0,\eta_0) = \int_0^{\xi_0} F^{(i)}(\xi,\eta_0)\, d\xi \geq 0, \quad i = 1, 2, \tag{3.17}$$

Assume now (**a**) $p_1^{(i)} \geq 0$ and $q_1^{(i)} \geq 0$ in $\bar{D}_\varepsilon^{(1)}$. Then, using (3.17), we find that $U_1^{(i)} \geq 0$. Thus, in both cases (**a**) or (**b**), the inequalities (3.15) - (3.17) hold. Suppose now that for some $m \in \mathbb{N}$

$$(U_m^{(i)} - U_{m-1}^{(i)}) \geq 0, \ (p_m^{(i)} - p_{m-1}^{(i)}) \geq 0, \ (q_m^{(i)} - q_{m-1}^{(i)}) \geq 0 \quad \text{in } \bar{D}_\varepsilon^{(1)}, \ i = 1, 2.$$

Then

$$E_m^{(i)} - E_{m-1}^{(i)} = A_i(p_m^{(i)} - p_{m-1}^{(i)}) + B_i(q_m^{(i)} - q_{m-1}^{(i)}) + C_i(U_m^{(i)} - U_{m-1}^{(i)})$$
$$+ D_i(U_m^{(i+1)} - U_{m-1}^{(i+1)}) \geq 0 \quad \text{in } \bar{D}_\varepsilon^{(1)}, \quad i = 1, 2,$$

where we denote $U_m^{(3)} := U_m^{(1)}$. Therefore, we see that

$$(U_{m+1}^{(i)} - U_m^{(i)})(\xi_0,\eta_0) = \int_0^{\xi_0} \int_{\xi_0}^{\eta_0} (E_m^{(i)} - E_{m-1}^{(i)})(\xi,\eta)\, d\eta\, d\xi$$
$$+2\int_0^{\xi_0} \int_0^{\eta} (E_m^{(i)} - E_{m-1}^{(i)})(\xi,\eta)\, d\xi\, d\eta$$
$$+\int_0^{\xi_0} \alpha(1-\xi)(U_m^{(i)} - U_{m-1}^{(i)})(\xi,\xi)\, d\xi \geq 0.$$

In the same manner,

$$(p_{m+1}^{(i)} - p_m^{(i)})(\xi_0, \eta_0) = \int_0^{\xi_0} (E_m^{(i)} - E_{m-1}^{(i)})(\xi, \xi_0)\, d\xi$$

$$+ \int_{\xi_0}^{\eta_0} (E_m^{(i)} - E_{m-1}^{(i)})(\xi_0, \eta)\, d\eta + \alpha(1-\xi_0)(U_m^{(i)} - U_{m-1}^{(i)})(\xi_0, \xi_0) \geq 0,$$

$$(q_{m+1}^{(i)} - q_m^{(i)})(\xi_0, \eta_0) = \int_0^{\xi_0} (E_m^{(i)} - E_{m-1}^{(i)})(\xi, \eta_0)\, d\xi \geq 0.$$

Finally, by induction, we conclude that

$$\begin{aligned} &U^{(i)}(\xi_0, \eta_0) = \textstyle\sum_{m=0}^{\infty} (U_{m+1}^{(i)} - U_m^{(i)})(\xi_0, \eta_0) \geq 0, \\ &p^{(i)}(\xi_0, \eta_0) \geq 0,\ q^{(i)}(\xi_0, \eta_0) \geq 0, \quad (\xi_0, \eta_0) \in \bar{D}_\varepsilon^{(1)}. \end{aligned} \tag{3.18}$$

■

REMARK 3.1. *Note here that Lemma 3.1 describes the behavior of the system* (2.14) *around the point* $(0,0,0)$. *This lemma becomes particularly useful in Sections 6 and 7 in finding singular solutions of the equation* (2.1). *None the less, when the equation* (2.1) *transforms to the system* (2.14), *by* (2.16), *we see that* $D_2 = -D_1 = na_2/4$. *Since, in view of Lemma 3.1,* $D_1 \geq 0$ *and* $D_2 \geq 0$, *it should be* $a_2 \equiv 0$. *Because of this fact, we are able to find singular solutions only in the cases where* $a_2 \equiv 0$ *(see also Introduction, Open questions, 3).*

As a consequence of Theorem 3.1 and representations (3.3) – (3.6), we have the following smoothness result:

THEOREM 3.2. *Let* $F^i, A_i, B_i, C_i, D_i \in C^1(\bar{D}_\varepsilon^{(1)})$, $i = 1, 2$, $\varepsilon > 0$. *Then there exists a classical solution* $U \in C^2(\bar{D}_\varepsilon^{(1)})$ *of Problem* $P_{\alpha,3}$.

Proof. Since we have already shown that

$$\begin{aligned} p_\eta^{(1)}(\xi_0, \eta_0) &\equiv q_\xi^{(1)}(\xi_0, \eta_0) \\ &= [F^1 + A_1 p^{(1)} + B_1 q^{(1)} + C_1 U^{(1)} + D_1 U^{(2)}](\xi_0, \eta_0) \end{aligned} \tag{3.19}$$

and that similar representations for $p_\eta^{(2)}$ and $q_\xi^{(2)}$ hold, we have to prove only that $p_\xi^{(i)}$ and $q_\eta^{(i)}$ exist and belong to $C(\bar{D}_\varepsilon^{(1)})$. Indeed:

1. For fixed η_0 the equality (3.19) is a linear ODE for the function $q^{(1)}(\xi_0, \eta_0)$. So, using the well known formula for the solution with the initial Cauchy data $q^{(1)}(0, \eta_0) = 0$ from (3.4), we find that

$$\begin{aligned} &q^{(1)}(\xi_0, \eta_0) = \\ &\int_0^{\xi_0} [F^1 + A_1 p^{(1)} + C_1 U^{(1)} + D_1 U^{(2)}](\xi, \eta_0) \exp\left(\int_\xi^{\xi_0} B_1(\tau, \eta_0) d\tau \right) d\xi. \end{aligned} \tag{3.20}$$

Since $F^1, A_1, B_1, C_1, D_1, U^{(1)}, U^{(2)} \in C^1(\bar{D}_\varepsilon^{(1)})$ and $p^{(1)}, p_\eta^{(1)} \in C(\bar{D}_\varepsilon^{(1)})$, by (3.20) we conclude that $q^{(1)} \in C^1(\bar{D}_\varepsilon^{(1)})$.

2. For fixed ξ_0 the equality (3.19) is a linear ODE for the function $p^{(1)}(\xi_0,\eta_0)$. So, arguments similar to above lead to

$$p^{(1)}(\xi_0,\eta_0)=G_1(\xi_0)\exp\left(\int_{\xi_0}^{\eta_0}A_1(\xi_0,\eta)d\eta\right) \tag{3.21}$$

$$+\int_{\xi_0}^{\eta_0}[F^1+B_1q^{(1)}+C_1U^{(1)}+D_1U^{(2)}](\xi_0,\eta)\exp\left(\int_{\eta}^{\eta_0}A_1(\xi_0,\tau)d\tau\right)d\eta$$

The function $G_1(\xi_0)$ follows from (3.3) and is

$$G_1(\xi_0)=\int_0^{\xi_0}[F^1+A_1p^{(1)}+B_1q^{(1)}+C_1U^{(1)}+D_1U^{(2)}](\xi,\xi_0)d\xi+\alpha(1-\xi_0)U^{(1)}(\xi_0,\xi_0).$$

Obviously $G_1\in C^1(\bar{D}_\varepsilon^{(1)})$, because F^1, A_1, B_1, C_1, D_1, α, $U^{(1)}$, $U^{(2)}$, $q^{(1)}\in C^1(\bar{D}_\varepsilon^{(1)})$ and $p^{(1)},p^{(1)}_\eta\in C(\bar{D}_\varepsilon^{(1)})$. Finally, from (3.21) it follows that $p^{(1)}\in C^1(\bar{D}_\varepsilon^{(1)})$. ■

REMARK 3.2. *Studying a solution of the Problem $P_{\alpha,3}$ in the domain $\bar{D}_\varepsilon^{(1)}$, actually we investigate the behavior of the solution of Problem $P_{\alpha,2}$ in the domain $\bar{D}_\delta$, when $\delta\to 0$, around the line $\eta=1$. It is easy to prove, that for any $\varepsilon\in(0,1)$ and $\delta\in(0,1)$ the solutions both of these problems coincide in their common domain $\bar{D}_\varepsilon^{(1)}\cap\bar{D}_\delta$.*

4. Existence and uniqueness theorems for $2-D$ Problem $\mathbf{P}_{\alpha,1}$

Consider now the $2-D$ problem

(4.1)

$$P_{\alpha,1}:\begin{cases}\frac{1}{\varrho}(\varrho u^{(1)}_\varrho)_\varrho-u^{(1)}_{tt}+a_1u^{(1)}_\varrho+bu^{(1)}_t+(c-\frac{n^2}{\varrho^2})u^{(1)}+na_2u^{(2)}=f^{(1)}\text{ in }G_\varepsilon,\\ \frac{1}{\varrho}(\varrho u^{(2)}_\varrho)_\varrho-u^{(2)}_{tt}+a_1u^{(2)}_\varrho+bu^{(2)}_t+(c-\frac{n^2}{\varrho^2})u^{(2)}-na_2u^{(1)}=f^{(2)}\text{ in }G_\varepsilon,\\ u^{(i)}|_{S_1\cap\partial G_\varepsilon}=0,\qquad [u^{(i)}_t+\alpha(\varrho)u^{(i)}]|_{S_0\cap\partial G_\varepsilon}=0\quad,i=1,2.\end{cases}$$

Note that, the notion of the generalized solution of the problem $P_{\alpha,1}$ in the domain G_ε , $\varepsilon\in(0,1)$, has been given by Definition 2.2.

THEOREM 4.1. *Let $a_1,a_2,b,c,f^{(1)},f^{(2)}\in C^1(\bar{G}_0\setminus(0,0))$. Then there exists a generalized solution $u=(u^{(1)},u^{(2)})\in C^2(\bar{G}_0\setminus(0,0))$ of problem $P_{\alpha,1}$ in G_0, which is a classical solution of the problem $P_{\alpha,1}$ in any domain G_ε, $\varepsilon\in(0,1)$.*

Proof. In view of (2.11) and (2.13), i.e. $z(\varrho,t)=\varrho^{1/2}u(\varrho,t)$ and $\xi=1-\varrho-t$, $\eta=1-\varrho+t$, introduce the function

$$U^{(i)}(\xi,\eta)=z^{(i)}(\varrho(\xi,\eta),t(\xi,\eta)).$$

Then Problem $P_{\alpha,1}$ in the new terms becomes $P_{\alpha,2}$, i.e.

$$\begin{cases}U^{(1)}_{\xi\eta}-A_1U^{(1)}_\xi-B_1U^{(1)}_\eta-C_1U^{(1)}-D_1U^{(2)}=F^1(\xi,\eta)\text{ in }D_\varepsilon,\\ U^{(2)}_{\xi\eta}-A_2U^{(2)}_\xi-B_2U^{(2)}_\eta-C_2U^{(2)}-D_2U^{(1)}=F^2(\xi,\eta)\text{ in }D_\varepsilon,\end{cases} \tag{4.2}$$

$$U^{(i)}(0,\eta)=0,(U^{(i)}_\eta-U^{(i)}_\xi)(\xi,\xi)+\alpha(1-\xi)U^{(i)}(\xi,\xi)=0,\quad i=1,2, \tag{4.3}$$

where the connection between the coefficients is given by (2.16). For each fixed $\varepsilon \in (0,1)$ Theorem 3.2 ensures the existence of a classical solution $(U^{(1)}, U^{(2)}) \in C^2(\bar{D}_\varepsilon^{(1)})$ of the problem $P_{\alpha,3}$. More precisely for any fixed $\varepsilon_1, \varepsilon_2$ with $0 < \varepsilon_1 < \varepsilon_2 < 1$ the corresponding vector-solution u_{ε_2} is a restriction of u_{ε_1} in the region D_{ε_2}. So, essentially we have a function of class $C^2(\bar{D}_0\backslash(0,0))$ which in any region D_ε coincides with the corresponding solution u_ε and is a classical solution of Problem $P_{\alpha,3}$. Remark that the inverse transformations (2.11) and (2.13) lead to a vector function $(u^{(1)}, u^{(2)}) \in C^2(\bar{G}_0\backslash(0,0))$ which is a classical solution of Problem $P_{\alpha,1}$ in each G_ε. This solution is also a generalized solution of the same problem in G_0, because for each concrete test function $v \in V_0$ there is an $\varepsilon_v > 0$ for which $v \equiv 0$ in $G_0\backslash G_\varepsilon$ and (1.8) coincides with (2.5).

The proof of the theorem is complete. ■

THEOREM 4.2. *Let $a_1, a_2, b, c \in C^1(\bar{G}_0 \setminus (0,0))$. Then for each fixed $\varepsilon \in (0,1)$ there exists at most one generalized solution of the problem $P_{\alpha,1}$ in G_ε.*

Proof. Let $(u_1^{(1)}, u_1^{(2)})$ and $(u_2^{(1)}, u_2^{(2)})$ are two generalized solutions of $P_{\alpha,1}$ in G_ε. Then for $u^{(i)} := u_1^{(i)} - u_2^{(i)}$, $i = 1, 2$, we see that

1) $u^{(i)} \in C^1(\bar{G}_\varepsilon)$, $[u_t^{(i)} + \alpha(\varrho)u^{(i)}]|_{S_0 \cap \partial G_\varepsilon} = 0$, $u^{(i)}|_{S_1 \cap \partial G_\varepsilon} = 0, i = 1, 2;$

2) the identities

(4.4)

$$\int_{G_\varepsilon} \left[u_t^{(1)}v_t^{(1)} - u_\varrho^{(1)}v_\varrho^{(1)} + \left(a_1u_\varrho^{(1)} + bu_t^{(1)} + (c - \frac{n^2}{\varrho^2})u^{(1)} + na_2u^{(2)}\right)v^{(1)}\right] \varrho d\varrho\, dt$$
$$= \int_{S_0 \cap \partial G_\varepsilon} \alpha(\varrho)u^{(1)}v^{(1)}\, \varrho\, d\varrho,$$
$$\int_{G_\varepsilon} \left[u_t^{(2)}v_t^{(2)} - u_\varrho^{(2)}v_\varrho^{(2)} + \left(a_1u_\varrho^{(2)} + bu_t^{(2)} + (c - \frac{n^2}{\varrho^2})u^{(2)} - na_2u^{(1)}\right)v^{(2)}\right] \varrho d\varrho\, dt$$
$$= \int_{S_0 \cap \partial G_\varepsilon} \alpha(\varrho)u^{(2)}v^{(2)}\, \varrho\, d\varrho$$

hold for all functions $v^{(1)}, v^{(2)} \in V_\varepsilon^{(1)}$. If the functions $v^{(i)} \in C^2(\bar{G}_\varepsilon)$, then from (4.4) we conclude that

(4.5)

$$\int_{G_\varepsilon} \left[\left(\frac{1}{\varrho}(\rho v_\varrho^{(1)})_\varrho - v_{tt}^{(1)} - \frac{1}{\varrho}(\varrho a_1 v^{(1)})_\varrho - (bv^{(1)})_t + (c - \frac{n^2}{\varrho^2})v^{(1)}\right)u^{(1)}\right.$$
$$\left. + na_2v^{(1)}u^{(2)}\right] \varrho d\varrho\, dt = 0,$$
$$\int_{G_\varepsilon} \left[\left(\frac{1}{\rho}(\rho v_\varrho^{(2)})_\varrho - v_{tt}^{(2)} - \frac{1}{\varrho}(\varrho a_1 v^{(2)})_\varrho - (bv^{(2)})_t + (c - \frac{n^2}{\varrho^2})v^{(2)}\right)u^{(2)}\right.$$
$$\left. - na_2v^{(2)}u^{(1)}\right] \varrho d\varrho\, dt = 0.$$

Let $h^{(1)}, h^{(2)} \in C^1(\bar{G}_0\backslash(0,0)$ be arbitrary. We are interested in the following

Problem $P^*_{\alpha,1}$. Find a solution $v^{(1)}, v^{(2)} \in V^{(1)}_\varepsilon \cap C^2(\bar{G}_\varepsilon)$ of the system

$$\frac{1}{\varrho}(\rho v^{(1)}_\varrho)_\varrho - v^{(1)}_{tt} - \frac{1}{\varrho}(\varrho a_1 v^{(1)})_\varrho - (bv^{(1)})_t + (c - \frac{n^2}{\varrho^2})v^{(1)} - na_2 v^{(2)} = h^{(1)},$$
$$\frac{1}{\varrho}(\varrho v^{(2)}_\varrho)_\varrho - v^{(2)}_{tt} - \frac{1}{\varrho}(\varrho a_1 v^{(2)})_\varrho - (bv^{(2)})_t + (c - \frac{n^2}{\varrho^2})v^{(2)} + na_2 v^{(1)} = h^{(2)}.$$

For $z^{(i)} = \varrho^{1/2} v^{(i)}$, $\xi_1 = 1 - \varepsilon - \eta$, $\eta_1 = 1 - \varepsilon - \xi$, and

$$V^{(i)}(\xi_1, \eta_1) = z^{(i)}(1 - \varepsilon - \eta_1, 1 - \varepsilon - \xi_1), \tag{4.6}$$

the domain G_ε maps to D_ε, and for appropriate coefficients A_i, B_i, C_i, D_i and $\beta = \alpha + b$ the above Problem $P^*_{\alpha,1}$ transforms to Darboux–Goursat Problem $P_{\beta,3}$. But for this problem Theorem 3.2 ensures the solvability in $C^2(\bar{D}_\varepsilon)$. Consequently, there exists a classical solution $(V^{(1)}, V^{(2)}) \in C^2(\bar{D}_\varepsilon)$ and so the inverse transformations (2.11) and (2.13) lead to a classical solution $(v^{(1)}, v^{(2)}) \in C^2(\bar{G}_\varepsilon)$ of Problem $P^*_{\alpha,1}$. Moreover, the system (4.5) becomes:

$$\begin{aligned} &\int_{G_\varepsilon} \left[\left(h^{(1)} + na_2 v^{(2)}\right) u^{(1)} + na_2 v^{(1)} u^{(2)}\right] \varrho d\varrho\, dt = 0, \\ &\int_{G_\varepsilon} \left[\left(h^{(2)} - na_2 v^{(1)}\right) u^{(2)} - na_2 v^{(2)} u^{(1)}\right] \varrho d\varrho\, dt = 0. \end{aligned} \tag{4.7}$$

Since the functions $h^{(1)}(\varrho,t), h^{(2)}(\varrho,t) \in C^1(\bar{G}_0 \setminus (0,0))$ are arbitrarily, (4.7) gives $u^{(1)}(\varrho,t) = u^{(2)}(\varrho,t) = 0$ in G_ε, i.e. $(u^{(1)}_1, u^{(2)}_1) \equiv (u^{(1)}_2, u^{(2)}_2)$. The proof is complete. ■

5. Existence and uniqueness theorems for $3-D$ Problem P_α

In this section we consider the following $3-D$ boundary value

Problem P_α. Find a solution of the equation

$$Lu = \frac{1}{\varrho}(\varrho u_\varrho)_\varrho + \frac{1}{\varrho^2}u_{\varphi\varphi} - u_{tt} + a_1 u_\varrho + a_2 u_\varphi + bu_t + cu = f(\varrho, \varphi, t) \text{ in } \Omega_\varepsilon, \tag{5.1}$$

which satisfies the boundary conditions

$$u\big|_{\Sigma_1 \cap \partial\Omega_\varepsilon} = 0, \quad [u_t + \alpha(\varrho)u]\big|_{\Sigma_0 \cap \partial\Omega_\varepsilon} = 0. \tag{5.2}$$

For this problem we state the following theorems.

THEOREM 5.1. *Let $a_1, a_2, b, c \in C^1(\bar{\Omega}_0 \setminus (0,0,0))$. Then for $0 \le \varepsilon < 1$ there exists at most one generalized solution of Problem P_α in Ω_ε.*

Proof. *Case* $0 < \varepsilon < 1$. If u_1, u_2 are two generalized solutions of P_α in Ω_ε, then $u := u_1 - u_2 \in C^1(\bar{\Omega}_\varepsilon)$ satisfies (5.2) and the identity

$$\begin{aligned} &\int_{\Omega_\varepsilon} [u_t v_t - u_\varrho v_\varrho - \frac{1}{\varrho^2} u_\varphi v_\varphi + (a_1 u_\varrho + a_2 u_\varphi + bu_t + cu)v] \varrho\, d\varrho\, d\varphi\, dt \\ &= \int_{\Sigma_0 \cap \partial\Omega_\varepsilon} \alpha(\varrho) uv \varrho\, d\varrho d\varphi \end{aligned} \tag{5.3}$$

holds for all $v \in V_\varepsilon$. We will show that in the Fourier expansion

$$u(\varrho, \varphi, t) = \sum_{n=0}^{\infty} \left\{u^{(1)}_n(\varrho, t) \cos n\varphi + u^{(2)}_n(\varrho, t) \sin n\varphi\right\} \tag{5.4}$$

the Fourier–coefficients $u_n^{(i)}(\varrho, t) \equiv 0$ in Ω_ε, $i = 1, 2$, i.e. $u \equiv 0$ in Ω_ε.

Since $u \in C^1(\bar{\Omega}_\varepsilon)$, using the substitution

$$v_1(\varrho, \varphi, t) = w_1(\varrho, t) \cos n\varphi \in V_\varepsilon \quad \text{or} \quad v_2(\varrho, \varphi, t) = w_2(\varrho, t) \sin n\varphi \in V_\varepsilon$$

in (5.3), we derive the system

$$\int\limits_{G_\varepsilon} \left[u_{n,t}^{(1)} w_{1,t} - u_{n,\varrho}^{(1)} w_{1,\varrho} + \left(a_1 u_{n,\varrho}^{(1)} + b u_{n,t}^{(1)} + (c - \frac{n^2}{\varrho^2}) u_n^{(1)} + n a_2 u_n^{(2)} \right) w_1 \right] \varrho d\varrho\, dt$$
$$= \int\limits_{S_0 \cap \partial G_\varepsilon} \alpha(\varrho) u_n^{(1)} w_1 \varrho\, d\varrho,$$

(5.5)

$$\int\limits_{G_\varepsilon} \left[u_{n,t}^{(2)} w_{2,t} - u_{n,\varrho}^{(2)} w_{2,\varrho} + \left(a_1 u_{n,\varrho}^{(2)} + b u_{n,t}^{(2)} + (c - \frac{n^2}{\varrho^2}) u_n^{(2)} - n a_2 u_n^{(1)} \right) w_2 \right] \varrho\, d\varrho dt$$
$$= \int\limits_{S_0 \cap \partial G_\varepsilon} \alpha(\varrho) u_n^{(2)} w_2 \varrho\, d\varrho,$$

for all $w_1, w_2 \in V_\varepsilon^{(1)}$ and $n \in \mathbb{N} \cup \{0\}$. By Definition 2.2 the function $(u_n^{(1)}, u_n^{(2)})(\varrho, t)$ is a generalized solution of the homogeneous problem $P_{\alpha,1}$. Clearly, Theorem 4.2 implies $u_n^{(1)}(\varrho, t) \equiv u_n^{(2)}(\varrho, t) \equiv 0$ in Ω_ε for $n \in \mathbb{N} \cup \{0\}$, and so $u^{(1)} = u_1 - u_2 \equiv 0$ in Ω_ε.

Case $\varepsilon = 0$. Let ε_0 be a fixed arbitrary number of $(0, 1)$. Then, by Lemma 2.1, it follows that the generalized solution $u \in C^1(\bar{\Omega}_0 \setminus (0, 0, 0))$ of Problem P_α in Ω_0 is also a generalized solution of the homogeneous problem P_α in Ω_{ε_0}. Since, by the previous case, $u \equiv 0$ in Ω_{ε_0} and $\varepsilon_0 > 0$ is arbitrary, we see that $u = u_1 - u_2 \equiv 0$ in Ω_0.

This completes the proof of the theorem. ∎

THEOREM 5.2. *Let $a_1, a_2, b, c \in C^1(\bar{\Omega}_0 \setminus (0, 0, 0))$ and the function $f \in C(\bar{\Omega}_0) \cap C^1(\bar{\Omega}_0 \setminus (0, 0, 0))$ be of the form:*

$$f(\varrho, \varphi, t) = \sum_{n=0}^{k} \left\{ f_n^{(1)}(\varrho, t) \cos n\varphi + f_n^{(2)}(\varrho, t) \sin n\varphi \right\},\ k \in \mathbb{N} \cup \{0\}. \tag{5.6}$$

Then there exists one and only one generalized solution

$$u(\varrho, \varphi, t) = \sum_{n=0}^{k} \left\{ u_n^{(1)}(\varrho, t) \cos n\varphi + u_n^{(2)}(\varrho, t) \sin n\varphi \right\} \tag{5.7}$$

of the problem P_α in Ω_0. This solution $u \in C^2(\bar{\Omega}_0 \setminus (0, ., 0))$ and is a classical solution of the problem P_α in each domain $\Omega_\varepsilon, \varepsilon \in (0, 1)$. Moreover, if

$$|a_1| \leq d\varrho^{-1}, |a_2| \leq d\varrho^{-2}, |b| \leq d\varrho^{-2}, |c| \leq d\varrho^{-2}, |\alpha| \leq d\varrho^{-2} \text{ in } \bar{\Omega}_0 \setminus (0, ., 0),$$

then for a fixed n the corresponding trigonometric polynomial u_n of degree n from (5.7) satisfies the following a priori estimates:

for $n=0$:

$$\begin{aligned}\|u_0(x_1,x_2,t)\|_{C^1(\tilde{\Omega}_\varepsilon)} &= \textstyle\sum_{|\alpha|\le 1}\sup_{\tilde{\Omega}_\varepsilon}|D^\alpha u_0| \\ &\le 6\varepsilon^{3/2}\exp\left(\frac{32d+2}{\varepsilon^2}\right)\|f_0^{(11)}\|_{C^0(\bar{G}_0)},\end{aligned} \tag{5.8}$$

while for $n\in\mathbb{N}$:

$$\begin{aligned}&\|u_n(x_1,x_2,t)\|_{C^1(\tilde{\Omega}_\varepsilon)} \\ &\le \frac{6\varepsilon^{3/2}}{n(n+2d)}\exp\left(\frac{8n(n+3d)}{\varepsilon^2}\right)\left(\|f_n^{(1)}\|_{C^0(\bar{G}_0)}+\|f_n^{(2)}\|_{C^0(\bar{G}_0)}\right),\end{aligned} \tag{5.9}$$

where $\Omega_\varepsilon^{(1)} = \Omega_0\cap\{(\varrho,t):\varrho+t>\varepsilon\}$.

Proof. It suffices to consider the case of a fixed number n. As in Section 2, we make the substitutions

$$\xi = 1-\varrho-t\ ,\ \eta = 1-\varrho+t, \tag{5.10}$$

and introduce the new function

$$U^{(i)}(\xi,\eta) = \varrho^{1/2}u^{(i)}(\varrho(\xi,\eta),t(\xi,\eta)). \tag{5.11}$$

If we denote

$$F^{(i)}(\xi,\eta) := \frac{1}{4\sqrt{2}}(2-\eta-\xi)^{1/2}f_n^{(i)}(\xi,\eta)\in C^1(\bar{D}_0\backslash(1,1)),$$

and use the notations of (2.16), the problem reduces to the Problem $P_{\alpha,3}$. Thus, we can use Theorems 3.1 and 3.2 to ensure the existence of a classical solution $(U^{(1)},U^{(2)})(\xi,\eta)$ of this problem with the estimates (3.8)

Case $n\in\mathbb{N}$. In view of (3.7), (2.16), it is easy to see that we can chose:

$$\begin{aligned} c(\varepsilon) &:= \frac{n(n+2d)}{\varepsilon^2},\quad M_\alpha := \frac{4d}{\varepsilon^2} \\ M &\le \frac{1}{4}\max\left\{\|f_n^{(1)}\|_{C^0(\bar{G}_0)},\|f_n^{(2)}\|_{C^0(\bar{G}_0)}\right\} := M_n,\end{aligned} \tag{5.12}$$

Hence, on one hand, Theorems 3.1 and 3.2 ensure the smoothness of the solution U of Problem $P_{\alpha,3}$ in $D_\varepsilon^{(1)} = \{(\xi,\eta): 0<\xi<\eta<1,\ 0<\xi<1-\varepsilon\}$, $\varepsilon>0.$, i.e.

$$(U_n^{(1)},U_n^{(2)})(\xi,\eta) := U\in C^2(\bar{D}_\varepsilon^{(1)}), \tag{5.13}$$

on the other hand, these theorems ensure the a priori estimates:

$$\begin{aligned}\sup_{D_\varepsilon^{(1)}}|U_n^{(i)}(\xi,\eta)| &\le M_n(4c(\varepsilon)+M_\alpha)^{-2}\exp(8c(\varepsilon)+2M_\alpha) \\ &\le M_n\varepsilon^4[4n(n+2d)]^{-2}\exp\left[(8n(n+3d)\varepsilon^{-2}\right],\end{aligned}$$

$$\sup_{D_\varepsilon^{(1)}}\{|U_{n,\xi}^{(i)}|,\ |U_{n,\eta}^{(i)}|\}\le M_n\varepsilon^2[4n(n+2d)]^{-1}\exp\left[(8n(n+3d)\varepsilon^{-2}\right].$$

Also, by (5.10) and (5.11), we have

$$u_n^{(i)}(\varrho,t) = \varrho^{-\frac{1}{2}}U_n^{(i)}(\xi,\eta)\,.$$

Since $\varrho \geq \varepsilon/2$ for $(\xi,\eta) \in D_\varepsilon^{(1)}$, by the inverse transformation we see that

$$
\begin{aligned}
|u_n^{(i)}(\varrho,t)| &\leq M_n \frac{\varepsilon^{7/2}}{8n^2(n+2d)^2} \exp\left(\frac{8n(n+3d)}{\varepsilon^2}\right), \\
|u_{n,t}^{(i)}(\varrho,t)| &\leq M_n \frac{\varepsilon^{3/2}}{n(n+2d)} \exp\left(\frac{8n(n+3d)}{\varepsilon^2}\right), \qquad (5.14)\\
|u_{n,\varrho}^{(i)}(\varrho,t)| &\leq M_n \frac{\varepsilon^{3/2}}{n(n+2d)} \exp\left(\frac{8n(n+3d)}{\varepsilon^2}\right).
\end{aligned}
$$

Therefore, in view of (5.7) and (5.14), for the trigonometrical polynomial

$$u_n(\varrho,\varphi,t) = u_n^{(1)}(\varrho,t)\cos n\varphi + u_n^{(2)}(\varrho,t)\sin n\varphi \qquad (5.15)$$

we derive

$$\|\frac{1}{\varrho}u_{n,\varphi}(\varrho,\varphi,t)\|_{C^\circ(\bar{\Omega}_\varepsilon)} \leq M_n \frac{\varepsilon^{5/2}}{4n(n+2d)^2} \exp\left(\frac{8n(n+3d)}{\varepsilon^2}\right). \qquad (5.16)$$

Since $u_n(\varrho\cos\varphi, \varrho\sin\varphi, t) = u_n^{(1)}(\varrho,\varphi,t)$, obviously

$$|u_{n,x_i}(x_1,x_2,t)| \leq 2M_n \frac{\varepsilon^{3/2}}{n(n+2d)} \exp\left(\frac{8n(n+3d)}{\varepsilon^2}\right), \quad i=1,2.$$

So, the estimate (5.9) holds in $\Omega_\varepsilon^{(1)}$.

Case $n=0$. In this case, by (5.6) and (5.7), it follows that $f_0(\varrho,\varphi,t) = f_0^{(1)}(\varrho,t)$ and $u_0(x_1,x_2,t) = u_0(\varrho,\varphi,t) = u_0^{(1)}(\varrho,t)$. Take

$$c(\varepsilon) := \frac{8d+1}{4\varepsilon^2}, \; M_\alpha := \frac{4d}{\varepsilon^2}, \; M := \frac{1}{4}\|f_0^{(1)}\|_{C^\circ(\bar{G}_0)}.$$

Then, arguments similar to the previous case lead to (5.8). ■

THEOREM 5.3. *Let the conditions of Theorem 5.2 be fulfilled. Also, for the sake of simplicity, suppose that* $a_1, a_2, b, c \in C^1(\bar{\Omega}_0)$ *and* $|\alpha'(\varrho)| \leq d_1/\varrho^{-3}$. *Then for a fixed* $n \in \mathbb{N}$ *the corresponding trigonometric polynomial* u_n *of degree* n *from* (5.15) *satisfies the following a priori estimate*

$$
\begin{aligned}
&\|u_n(x_1,x_2,t)\|_{C^2(\bar{\Omega}_\varepsilon^{(1)})} \\
&\leq C_1\varepsilon^{-1/2}\exp\left(\frac{8n(n+3d)}{\varepsilon^2}\right)\left(\|f_n^{(1)}\|_{C^\circ(\bar{G}_0)} + \|f_n^{(2)}\|_{C^\circ(\bar{G}_0)}\right), \qquad (5.17)
\end{aligned}
$$

where the constant C_1 *does not depend on* n *and* ε.

Proof. We will use the estimates of Theorem 5.2 and the representations of the second derivatives of Theorem 3.2. Following the same arguments as in Theorem 5.2, we obtain the estimates:

$$\sup_{D_\varepsilon^{(1)}}\{|U_{n,\xi\eta}^{(i)}|, |U_{n,\eta\eta}^{(i)}|, \; |U_{n,\xi\xi}^{(i)}|\} \leq C_1 M_n \exp\left[(8n(n+3d)\varepsilon^{-2}\right], \; i=1,2$$

and finally conclude that

$$\sup_{D_\varepsilon^{(1)}}\{|u_{n,x_ix_j}|, |u_{n,tx_i}|\} \leq C_1 M_n \varepsilon^{-1/2}\exp\left(\frac{8n(n+3d)}{\varepsilon^2}\right). \qquad ■$$

The next theorem is an immediate consequence of Theorems 5.1, 5.2 and 5.3.

THEOREM 5.4. *Let $f \in C^1(\bar{\Omega}_0)$ be of the form*

$$f(\varrho,\varphi,t) = \sum_{n=0}^{\infty}\{f_n^{(1)}(\varrho,t)\cos n\varphi + f_n^{(2)}(\varrho,t)\sin n\varphi\}. \tag{5.18}$$

Suppose that the Fourier coefficients $f_n^{(1)}(\varrho,t)$ and $f_n^{(2)}(\varrho,t)$ satisfy

(5.19)

$$\|f\|_{\exp(\varepsilon)} := \varepsilon^{3/2}\exp\left(\frac{32d+2}{\varepsilon^2}\right)\|f_0^{(1)}\|_{C^o(\bar{G}_0)}$$
$$+\sum_{n=1}^{\infty}\frac{\varepsilon^{3/2}}{n(n+2d)}\exp\left(\frac{8n(n+3d)}{\varepsilon^2}\right)\left(\|f_n^{(1)}\|_{C^o(\bar{G}_0)} + \|f_n^{(2)}\|_{C^o(\bar{G}_0)}\right) < \infty\,.$$

Then there exists one and only one generalized solution $u \in C^1(\bar{\Omega}_\varepsilon^{(1)})$ of the problem P_α in Ω_ε and the a priori estimate

$$\|u\|_{C^1(\bar{\Omega}_\varepsilon^{(1)})} \le 6\|f\|_{\exp(\varepsilon)} \tag{5.20}$$

holds. Moreover, if

$$\|f\|_{\exp_1(\varepsilon)} := \varepsilon^{-1/2}\left\{\exp\left(\frac{32d+2}{\varepsilon^2}\right)\|f_0^{(1)}\|_{C^o(\bar{G}_0)}\right.$$
$$\left.+\sum_{n=1}^{\infty}\exp\left(\frac{8n(n+3d)}{\varepsilon^2}\right)\left(\|f_n^{(1)}\|_{C^o(\bar{G}_0)} + \|f_n^{(2)}\|_{C^o(\bar{G}_0)}\right)\right\} < \infty, \tag{5.21}$$

then $u \in C^2(\bar{\Omega}_\varepsilon^{(1)})$, it is a classical solution of the problem P_α in Ω_ε and satisfies the a priori estimate

$$\|u\|_{C^2(\bar{\Omega}_\varepsilon^{(1)})} \le C_2\|f\|_{\exp_1(\varepsilon)}. \tag{5.22}$$

REMARK 5.1. *It is obvious that the estimates* (5.20) *or* (5.22) *hold if the series* (5.19) *and* (5.21) *are finite. In this case we have a solution which is of class $C^1(\bar{\Omega}_0\backslash(0,0,0))$ or of class $C^2(\bar{\Omega}_0\backslash(0,0,0))$. For example, the condition* (5.21) *is valid for each $\varepsilon \in (0,1)$, if there exists a function ψ with $\psi(n) \to +\infty$ as $n \to +\infty$ such that*

$$\sum_{n=1}^{\infty}\exp\left(8n(n+3d)\psi(n)\right)\left(\|f_n^{(1)}\|_{C^o(\bar{G}_0)} + \|f_n^{(2)}\|_{C^o(\bar{G}_0)}\right) < \infty. \quad \blacksquare \tag{5.23}$$

6. On the singularity of solutions of Problem P_α

In the case of the equation

$$Lu = \frac{1}{\varrho}(\varrho u_\varrho)_\varrho + \frac{1}{\varrho^2}u_{\varphi\varphi} - u_{tt} + a_1u_\varrho + a_2u_\varphi + bu_t + cu = f(\varrho,\varphi,t) \text{ in } \Omega_0, \tag{6.1}$$

consider the boundary conditions of Problem P_α, i.e.

$$P_\alpha: \quad u\big|_{\Sigma_1} = 0, \qquad [u_t + \alpha(\varrho)u]\big|_{\Sigma_0} = 0 \tag{6.2}$$

and prove the following interesting

THEOREM 6.1. *Let $\alpha(\varrho) \geq 0$, $\alpha(\varrho) \in C^1([0,1])$; a_1, b, $c \in C^1(\bar{\Omega}_0)$, $a_2 \equiv 0$ and*

$$a_1(\varrho, t) \geq |b|(\varrho, t), \quad a_1(\varrho, t) \geq 2\varrho c(\varrho, t), \quad (\varrho, t) \in \bar{\Omega}_0. \tag{6.3}$$

Then for each $n \in \mathbb{N}$ there exists a function $f_n(\varrho, \varphi, t) \in C^{n-2}(\bar{\Omega}_0)$ for which the corresponding generalized solution u_n of the problem P_α belongs to $C^2(\bar{\Omega}_0 \backslash (0,0,0))$ and the estimate

$$|u_n(\varrho, \varphi, \varrho)| \geq \frac{1}{2}|u_n(2\varrho, \varphi, 0)| + \varrho^{-n}|\cos n\varphi| \geq \varrho^{-n}|\cos n\varphi|, \qquad 0 < \varrho < 1, \tag{6.4}$$

holds.

Proof. Note that, by Theorem 1.1, the functions of $C^{n-2}(\bar{\Omega}_0)$

$$w_n(\varrho, \varphi, t) = \varrho^{-n}(\varrho^2 - t^2)^{n-1/2}(a_n \cos n\varphi + b_n \sin n\varphi), n \geq 4,$$

are classical solutions of the homogeneous Problem P^*_α for the wave equation with $\alpha \equiv 0$.

Consider the special case of Problem P_α :

$$Lu = \varrho^{-n}(\varrho^2 - t^2)^{n-1/2} \cos n\varphi \quad \text{ in } \Omega_0. \tag{6.5}$$

Theorem 5.1 declares that the equation (6.5) with boundary conditions (6.2) has at most one generalized solution. On the other hand, from Theorem 5.2 it is known that for the above right-hand side there exists a generalized solution in Ω_0 of the form

$$u_n(\varrho, \varphi, t) = u_n^{(1)}(\varrho, t) \cos n\varphi \in C^2(\bar{\Omega}_0 \backslash (0,0,0)),$$

which is classical solution in Ω_ε, $\varepsilon \in (0,1)$. By setting $u_n^{(2)}(\varrho, t) = \varrho^{\frac{1}{2}} u_n^{(1)}(\varrho, t)$ and substituting

$$\xi = 1 - \varrho - t, \qquad \eta = 1 - \varrho + t, \tag{6.6}$$

the equation (6.5), with boundary conditions (6.2), in view of

$$U(\xi, \eta) = u_n^{(2)}(\varrho(\xi, \eta), t(\xi, \eta)), \tag{6.7}$$

becomes a Darboux-Goursat problem $P_{\alpha,3}$:

$$U_{\xi\eta} - AU_\xi - BU_\eta - CU = F(\xi, \eta), \tag{6.8}$$

$$U(0, \eta) = 0, \qquad [(U_\eta - U_\xi) + \alpha(1-\xi)U_n]\big|_{\eta=\xi} = 0. \tag{6.9}$$

Note here that, because of the condition $a_2 \equiv 0$ and the special right-hand side of (6.5), we obtain no more a system as in the general case of section 3, but a single equation (6.8). According to (2.16), the coefficients of (6.8) are defined as follows:

$$\begin{gathered} A = \tfrac{1}{4}(a_1 + b) \geq 0, \quad B = \tfrac{1}{4}(a_1 - b) \geq 0, \\ C(\xi, \eta) = \frac{1}{4}\left(\frac{4n^2 - 1}{(2 - \eta - \xi)^2} + \frac{a_1(\xi, \eta)}{(2 - \eta - \xi)} - c(\xi, \eta)\right) \geq 0, \ n \in \mathbb{N}, \end{gathered} \tag{6.10}$$

$$F(\xi, \eta) = 2^{n-\frac{5}{2}} \left[\frac{(1-\xi)(1-\eta)}{2 - \eta - \xi}\right]^{n-\frac{1}{2}} \in C^{n-1}(\bar{D}_\varepsilon^{(1)}), \quad F(\xi, \eta) \geq 0, \tag{6.11}$$

where we preserve the same notations for a_1, b and c in the new coordinates (ξ, η). Next, in view of Theorem 3.2 and Lemma 3.1, we formulate the following result.

Proposition 6.1. *There exists a classical solution $U(\xi,\eta) \in C^2(\bar{D}_0 \setminus (1,1))$ for the problem (6.8), (6.9) for which*

$$U(\xi,\eta) \geq 0, \quad U_\xi(\xi,\eta) \geq 0, \quad U_\eta(\xi,\eta) \geq 0 \text{ in } \bar{D}^{(1)}_\varepsilon.$$

Set

$$K = \int_{D^{(1)}_{\frac{1}{2}}} F^2(\xi,\eta)\, d\eta\, d\xi > 0. \tag{6.12}$$

Then from (6.8) for $0 < \varepsilon < 1/2$ it follows that

$$\begin{aligned} 0 < K &\leq \int_{D^{(1)}_\varepsilon} F^2(\xi,\eta)\, d\eta\, d\xi = \int_{D^{(1)}_\varepsilon} (U_{\xi\eta}F)(\xi,\eta)\, d\eta\, d\xi \\ &- \int_{D^{(1)}_\varepsilon} [(AU_\xi + BU_\eta + CU)F](\xi,\eta)\, d\eta\, d\xi =: I_1 - I_2. \end{aligned} \tag{6.13}$$

Using the properties of $F(\xi,\eta)$ from (6.11) and following [**9**] we find that

(6.14)

$$\begin{aligned} I_1 := \int_0^{1-\varepsilon} \int_\xi^1 (U_{\xi\eta}F)(\xi,\eta)\, d\eta\, d\xi = \int_{D^{(1)}_\varepsilon} (UF_{\xi\eta})(\xi,\eta)\, d\xi\, d\eta \\ - \int_0^{1-\varepsilon} [U_\xi(\xi,\xi)F(\xi,\xi) + U(\xi,\xi)F_\eta(\xi,\xi)]\, d\xi - \int_{1-\varepsilon}^1 U(1-\varepsilon,\eta)F_\eta(1-\varepsilon,\eta)\, d\eta. \end{aligned}$$

An elementary calculation shows that

$$F_\xi(\xi,\eta) \leq 0, F_\eta(\xi,\eta) \leq 0, \tag{6.15}$$

(which actually follows from Lemma 3.1) and

$$F_\xi(\xi,\xi) = F_\eta(\xi,\xi) = \frac{1}{16}(1-2n)(1-\xi)^{n-\frac{3}{2}}. \tag{6.16}$$

>From (6.13) and (6.14) we see that

(6.17)

$$\begin{aligned} 0 < K \leq I_1 - I_2 = -\int_0^{1-\varepsilon} [U_\xi(\xi,\xi)F(\xi,\xi) + U(\xi,\xi)F_\xi(\xi,\xi)]\, d\xi \\ - \int_{1-\varepsilon}^1 U(1-\varepsilon,\eta)F_\eta(1-\varepsilon,\eta)\, d\eta + \int_{D^{(1)}_\varepsilon} \{(F_{\xi\eta} - CF)U - F(AU_\xi + BU_\eta)\}\, d\xi\, d\eta. \end{aligned}$$

Also, it is easy to check that

$$F_{\xi\eta}(\xi,\eta) - \frac{4n^2-1}{4(2-\eta-\xi)^2} F(\xi,\eta) = 0 \tag{6.18}$$

and so, because of (6.3), (6.10) and Proposition 6.1

$$(F_{\xi\eta} - CF)U - F(AU_\xi + BU_\eta) = -\left(\frac{a_1(\xi,\eta)}{(2-\eta-\xi)} - c(\xi,\eta)\right)\frac{FU}{4} - F(AU_\xi + BU_\eta) \leq 0.$$

Thus, we find

$$
\begin{aligned}
0 < K \leq I_1 - I_2 \leq & - \int_0^{1-\varepsilon} [U_\xi(\xi,\xi)F(\xi,\xi) + U(\xi,\xi)F_\xi(\xi,\xi)]\, d\xi \\
& - \int_{1-\varepsilon}^{1} U(1-\varepsilon,\eta) g_\eta(1-\varepsilon,\eta)\, d\eta,
\end{aligned} \tag{6.19}
$$

where, as it is easy to check

$$
F_\xi(\xi,\xi) = \frac{1}{2}[F(\xi,\xi)]_\xi. \tag{6.20}
$$

The function $U(\xi,\eta)$ is a classical solution of (6.8), (6.9) in $\bar{D}_\varepsilon$, $\varepsilon \in (0,1)$ with

$$
U_\xi(\xi,\xi) = \frac{1}{2}[U(\xi,\xi)]_\xi + \frac{1}{2}\alpha(1-\xi)U(\xi,\xi). \tag{6.21}
$$

If we substitute (6.20) and (6.21) into (6.19), we get

$$
\begin{aligned}
& K \leq I_1 - I_2 = -\frac{1}{2}\int_0^{1-\varepsilon} [F(\xi,\xi)U(\xi,\xi)]_\xi\, d\xi \\
& -\frac{1}{2}\int_0^{1-\varepsilon} \alpha(1-\xi)U(\xi,\xi)F(\xi,\xi)\, d\xi - \int_{1-\varepsilon}^{1} U(1-\varepsilon,\eta)F_\eta(1-\varepsilon,\eta)\, d\eta \\
& = -\frac{1}{2}(FU)(1-\varepsilon,1-\varepsilon) - \frac{1}{2}\int_0^{1-\varepsilon} \alpha(1-\xi)U(\xi,\xi)F(\xi,\xi)\, d\xi \\
& - \int_{1-\varepsilon}^{1} U(1-\varepsilon,\eta)F_\eta(1-\varepsilon,\eta)\, d\eta.
\end{aligned} \tag{6.22}
$$

In view of Proposition 6.1 and the properties of the function $F(\xi,\eta)$, we find

$$
U(\xi,\eta) \geq 0,\ U_\eta(\xi,\eta) \geq 0,\ \alpha(\xi) \geq 0,\ F(\xi,\eta) \geq 0,\ F_\eta(\xi,\eta) \leq 0 \text{ in } \bar{D}_\varepsilon^{(1)},
$$

which together with (6.22) and because of $F(1-\varepsilon,1) = 0$ implies

$$
\begin{aligned}
K \leq I_1 + I_2 & \leq -\int_{1-\varepsilon}^{1} U(1-\varepsilon,\eta)F_\eta(1-\varepsilon,\eta)\, d\eta - \frac{1}{2}(FU)(1-\varepsilon,1-\varepsilon) \\
& = \int_{1-\varepsilon}^{1} U(1-\varepsilon,\eta)|F_\eta(1-\varepsilon,\eta)|\, d\eta - \frac{1}{2}(FU)(1-\varepsilon,1-\varepsilon) \\
& \leq \int_{1-\varepsilon}^{1} U(1-\varepsilon,1)|F_\eta(1-\varepsilon,\eta)|\, d\eta - \frac{1}{2}(FU)(1-\varepsilon,1-\varepsilon) \\
& = \left[U(1-\varepsilon,1) - \frac{1}{2}U(1-\varepsilon,1-\varepsilon)\right] F(1-\varepsilon,1-\varepsilon).
\end{aligned}
$$

Since $F(1-\varepsilon,1-\varepsilon) = \frac{1}{4}\varepsilon^{n-\frac{1}{2}}$, we see that

$$
0 < K \leq \left[U(1-\varepsilon,1) - \frac{1}{2}U(1-\varepsilon,1-\varepsilon)\right] \frac{1}{4}\varepsilon^{n-\frac{1}{2}}.
$$

For $\xi = 1-\varepsilon$, $\eta = 1$ we have $\varrho = t = \varepsilon/2$ and so

$$0 < 4K\varepsilon^{\frac{1}{2}-n} \le u_n^{(2)}\left(\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right) - \frac{1}{2}u_n^{(2)}(\varepsilon,0). \tag{6.23}$$

Finally, the inverse transformation gives

$$u_n^{(1)}\left(\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right) \ge \frac{1}{2}u_n^{(1)}(\varepsilon,0) + \tilde{C}_1\varepsilon^{-n} \ge \tilde{C}_1\varepsilon^{-n},\ 0<\varepsilon<\frac{1}{2},$$

with $\tilde{C}_1 = 2^{\frac{5}{2}}K$. Multiplying the function u_n by $\tilde{C}_1^{-1}$, we see that (6.4) holds.

The proof is complete. ■

Note that the conditions of Theorem 6.1 are only sufficient ones, but not invariant with respect to change of variables. Now, we use this fact in order to find some new singular solutions. For this reason, consider the special form of the equation (6.1)

$$Lu = \frac{1}{\varrho}(\varrho u_\varrho)_\varrho + \frac{1}{\varrho^2}u_{\varphi\varphi} - u_{tt} + a_1(u_\varrho - u_t) + cu = f(\varrho,\varphi,t) \text{ in } \Omega_\varepsilon, \tag{6.24}$$

with the boundary conditions (6.2)

THEOREM 6.2. *Let a_1, $c \in C^1(\bar\Omega_0)$, $a_1(\varrho,t) \ge 2\varrho c(\varrho,t)$ for $(\varrho,t)\in\bar\Omega_0$ and $\alpha(\varrho)\in C^1([0,1])$, but without any conditions on the sign. Then the conclusion of Theorem 6.1 is satisfied for the equation* (6.24).

Proof. We are looking for an appropriate right-hand side functions f_n of the equation (6.24), for which some singular solutions exist. Set

$$u(\varrho,\varphi,t) = \exp(\lambda(\varrho+t))w(\varrho,\varphi,t),$$

where the function $\lambda(s)$ will be chosen later. Then the equation (6.24) becomes

$$\begin{aligned}(6.25)\,L_1w &= \frac{1}{\varrho}(\varrho w_\varrho)_\varrho + \frac{1}{\varrho^2}w_{\varphi\varphi} - w_{tt} + (a_1+2\lambda')(w_\varrho - w_t) + (c+\lambda'\varrho^{-1})w\\ &= \exp(-\lambda(\varrho+t))f(\varrho,\varphi,t) \text{ in } \Omega_\varepsilon\end{aligned}$$

and so we lead to the following boundary value problem:

$$P_\beta : L_1w = g = \exp(-\lambda(\varrho+t))f \text{ in } \Omega_0, u\big|_{\Sigma_1} = 0, [u_t+\beta(\varrho)u]\big|_{\Sigma_0} = 0 \tag{6.26}$$

with $\beta(\varrho) = \alpha(\varrho)+\lambda'(\varrho)$. To apply Theorem 6.1 to Problem P_β we need the following conditions

$$\alpha(\varrho)+\lambda'(\varrho)\ge 0,\quad a_1(\varrho,t)+2\lambda'(\varrho)\ge 0,\quad a_1(\varrho,t)-2\varrho c(\varrho,t)\ge 0,$$

which are satisfied, for example, for $\lambda(\varrho) = \Lambda\varrho$ and $\Lambda > 0$ large enough. Following the proof of Theorem 6.1 and using the transformations (6.6) and (6.7), we become the function $W(\xi,\eta)$ of (6.7), for which the equation (6.8) reduces to

$$W_{\xi\eta} - \frac{1}{4}(a_1+2\lambda')W_\eta - CW = F(\xi,\eta). \tag{6.27}$$

Here $C(\xi,\eta)$ and $F(\xi,\eta)$ are functions from (6.11) *and* (6.10).

Formulate now

Proposition 6.2. *There exists a classical solution $W(\xi,\eta)\in C^2(\bar D_0\setminus(1,1))$ of the problem* (6.27), (6.9) *for which*

$$W(\xi,\eta)\ge 0,\quad W_\xi(\xi,\eta)\ge 0,\quad W_\eta(\xi,\eta)\ge 0 \text{ in } \bar D_\varepsilon^{(1)}. \tag{6.28}$$

For the function

$$g_n(\varrho,\varphi,t) = \varrho^{-n}(\varrho^2 - t^2)^{n-1/2}\cos n\varphi, \tag{6.29}$$

as the right-hand side of the equation $L_1 w = g$, the Theorem 6.1 gives a singular solution w_n with the estimates (6.4). That is,

$$|w_n(\varrho,\varphi,\varrho)| \geq \frac{1}{2}|w_n(2\varrho,\varphi,0)| + \varrho^{-n}|\cos n\varphi| \geq \varrho^{-n}|\cos n\varphi|,\ 0<\varrho<1. \tag{6.30}$$

Now, the inverse transform $u_n = \exp(\Lambda(\varrho+t))w_n$ gives

$$|u_n(\varrho,\varphi,\varrho)| \geq \frac{1}{2}|u_n(2\varrho,\varphi,0)| + \varrho^{-n}|\cos n\varphi| \geq \varrho^{-n}|\cos n\varphi|,\ 0<\varrho<1, \tag{6.31}$$

where the function $u_n(\varrho,\varphi,t)$ is a solution of the problem (6.24), (6.2) with

$$f(\varrho,\varphi,t) = \exp(\Lambda(\varrho+t))\ \varrho^{-n}(\varrho^2-t^2)^{n-1/2}\cos n\varphi.$$

The proof is complete. ■

REMARK 6.1. *Aldashev in* [**2**] *considers the equation* (2.1) *and studies the homogeneous Problems* P_α *and* P_α^*. *Unfortunately, as it easy to check, the procedure which he follows leads to correct conclusion only in the case of wave equation, i.e. only in the case when all the lower order terms in* (2.1) *are identically zero. Otherwise this procedure leads to systems of differential equations which are nonequivalent to such ones (see* (2.7)*) which should be solved. This is due to the fact that, in the systems obtained in* [**2**] *by integration with respect to* φ, *the Fourier coefficients* u_k *of degree* k *depend on such ones* u_{k-1} *of degree* $k-1$.

7. Applications to the wave equation, singular solutions

In this section we consider the wave equation

$$\Box u = \frac{1}{\varrho}(\varrho u_\varrho)_\varrho + \frac{1}{\varrho^2}u_{\varphi\varphi} - u_{tt} = f(\varrho,\varphi,t) \tag{7.1}$$

subject to the boundary value problem P_α, i.e.

$$P_\alpha: \quad \Box u = f \text{ in } \Omega_0, \quad u\big|_{\Sigma_1} = 0, \quad [u_t + \alpha(\varrho)u]\big|_{\Sigma_0} = 0. \tag{7.2}$$

As an application to the wave equation of the results of the previous section, we formulate the following

THEOREM 7.1. *Let* $\alpha(\varrho) \in C^\infty([0,1])$ *be an arbitrary function. Then:*
(i) *For each* $n \in \mathbb{N}$, $n \geq 4$, *there exists a function* $f_n(\varrho,\varphi,t) \in C^{n-2}(\bar{\Omega}_0)$, *for which the corresponding generalized solution* u_n *of the problem* P_α *belongs to* $C^n(\bar{\Omega}_0\backslash(0,0,0))$ *and satisfies the estimate*

$$|u_n(\varrho,\varphi,\varrho)| \geq \frac{1}{2}|u_n(2\varrho,\varphi,0)| + \varrho^{-n}|\cos n\varphi| \geq \varrho^{-n}|\cos n\varphi|, \qquad 0<\varrho<1, \tag{7.3}$$

(ii). *In the case* $\alpha(\varrho) \leq 0$ *the upper estimate of the singular solution* u_n *is*

$$|u_n(\varrho,\varphi,t)| \leq c_\mu \varrho^{-1/2}\left(\frac{\varrho}{(\varrho+t)(\varrho-t)}\right)^{n-\frac{1}{2}}|\cos n\varphi|, \quad (\varrho,t)\in D_1^\mu \tag{7.4}$$

where $c_\mu = const$ *and*

$$D_1^\mu := \{(\varrho,t) : 0 < \varrho - t \leq \varrho + t \leq \mu(\varrho - t)\}, \mu < 2^{\frac{2n+1}{2n-1}} - 1.$$

Thus, for $\alpha(\varrho) \leq 0$ we have two-sided estimates, which in the limit cases $t = \varrho$ and $t = 0$ are:

$$\varrho^{-n}|\cos n\varphi| \leq |u_n(\varrho, \varphi, \varrho)| \quad \text{and} \quad |u_n(\varrho, \varphi, 0)| \leq C\varrho^{-n}|\cos n\varphi|, \tag{7.5}$$

with $C = const$. That is, in the case of $\alpha(\varrho) \leq 0$ the exact behavior of $u_n(x_1, x_2, t)$ around (0,0,0) is $(x_1^2 + x_2^2)^{-n/2}$.

Proof. Note that, the wave equation (7.1) is of the form (6.24) and so the first part of Theorem 7.1 follows from Theorem 6.2 and [**9**]. Actually, according to Theorem 6.2 we choice the function f_n to be of the following special form:

$$\Box u = f_n = \exp(\Lambda(\varrho + t))\ \varrho^{-n}(\varrho^2 - t^2)^{n-1/2} \cos n\varphi \qquad \text{in } \Omega_0, \tag{7.6}$$

where $\Lambda > 0$ is large enough and such that $\Lambda + \alpha(\varrho) \geq 0$, $\varrho \in [0, 1]$. Then by Theorems 5.1 and 5.2 there exists a unique generalized solution $u_n(\varrho, \varphi, t)$ of equation (7.6) satisfying the boundary conditions (7.2) and the estimates (7.3) (see Theorem 6.2). On the other hand, by Theorem 5.2 [**9**], for the equation (7.6) there exists a generalized solution in Ω_0 of the form

$$u_n(\varrho, \varphi, t) = u_n^{(1)}(\varrho, t) \cos n\varphi \in C^n(\bar{\Omega}_0 \backslash (0, 0, 0)),$$

which is a classical solution of Problem P_α in Ω_ε, $\varepsilon \in (0, 1)$.

(ii) By setting $u_n^{(2)}(\varrho, t) = \varrho^{\frac{1}{2}} u_n^{(1)}(\varrho, t)$ and substituting

$$\xi = 1 - \varrho - t, \qquad \eta = 1 - \varrho + t, \tag{7.7}$$

the problem (7.6), (7.2), in view of

$$U_n(\xi, \eta) = u_n^{(2)}(\varrho(\xi, \eta), t(\xi, \eta)), \tag{7.8}$$

becomes a Darboux-Goursat problem $P_{\alpha,3}$:

$$U_{n,\xi\eta} - C(\xi, \eta)U_n = G(\xi, \eta) \equiv \exp(\Lambda(1 - \xi))F(\xi, \eta), \tag{7.9}$$

$$U_n(0, \eta) = 0, \qquad [U_{n,\eta} - U_{n,\xi} + \alpha(1 - \xi)U_n]\big|_{\eta=\xi} = 0. \tag{7.10}$$

Here, the coefficients

$$C(\xi, \eta) = \frac{4n^2 - 1}{4(2 - \eta - \xi)^2} \in C^\infty(\bar{D}_\varepsilon^{(1)}), \quad n \geq 4, \tag{7.11}$$

and

$$F(\xi, \eta) = 2^{n-\frac{5}{2}} \left[\frac{(1 - \xi)(1 - \eta)}{2 - \eta - \xi}\right]^{n-\frac{1}{2}} \in C^{n-1}(\bar{D}_\varepsilon^{(1)}) \tag{7.12}$$

are defined by (2.16) and (2.15). We need now some information on the behavior of the function $U_n(\xi, \eta)$. Since, by Theorem 6.2,

$$U_n(\xi, \eta) = \exp(\Lambda(\varrho + t))\ W(\xi, \eta) = \exp(\Lambda(1 - \xi))\ W(\xi, \eta),$$

$W(\xi, \eta) \geq 0$ and $W_\eta(\xi, \eta) \geq 0$ in $\bar{D}_\varepsilon^{(1)}$, we formulate the following result.

Proposition 7.1. *There exists a classical solution $U_n(\xi, \eta) \in C^n(\bar{D}_0 \setminus (1, 1))$ for the problem (6.8), (6.9) for which*

$$U_n(\xi, \eta) \geq 0, \quad U_{n,\eta}(\xi, \eta) \geq 0, \quad (\xi, \eta) \in \bar{D}_\varepsilon^{(1)}. \tag{7.13}$$

Set

$$K_1 = \int_{D_0} G^2(\xi,\eta)\,d\xi\,d\eta > 0. \tag{7.14}$$

Then by (7.9), for $0 < \varepsilon < 1$ it follows that

$$\begin{aligned} K_1 &\geq \int_{D_\varepsilon^{(1)}} G^2(\xi,\eta)\,d\xi\,d\eta \geq \int_{D_\varepsilon^{(1)}} G(\xi,\eta)\,F(\xi,\eta)\,d\xi\,d\eta \\ &= \int_{D_\varepsilon^{(1)}} U_{n,\xi\eta}F(\xi,\eta)\,d\xi\,d\eta - \int_{D_\varepsilon^{(1)}} C(\xi,\eta)U_n(\xi,\eta)F(\xi,\eta)\,d\xi\,d\eta =: I_1 - I_2, \end{aligned} \tag{7.15}$$

where

$$\begin{aligned} I_1 &= \int_0^{1-\varepsilon}\int_\xi^1 (U_{n,\xi\eta}F)(\xi,\eta)\,d\eta\,d\xi \\ &= -\int_0^{1-\varepsilon} U_{n,\xi}(\xi,\xi)F(\xi,\xi)\,d\xi - \int_{D_\varepsilon^{(1)}} (U_{n,\xi}F_\eta)(\xi,\eta)\,d\xi\,d\eta, \end{aligned} \tag{7.16}$$

and by (7.12), $F(\xi,1) = 0$. Since

$$\begin{aligned} \int_{D_\varepsilon^{(1)}} (U_{n,\xi}F_\eta)(\xi,\eta)\,d\xi\,d\eta &= \int_0^{1-\varepsilon} (U_nF_\eta)(\eta,\eta)\,d\eta \\ &+ \int_{1-\varepsilon}^1 (U_nF_\eta)(1-\varepsilon,\eta)\,d\eta - \int_{D_\varepsilon^{(1)}} (U_nF_{\xi\eta})(\xi,\eta)\,d\xi\,d\eta, \end{aligned} \tag{7.17}$$

(7.16) becomes

$$\begin{aligned} I_1 = &-\int_0^{1-\varepsilon} [U_{n,\xi}(\xi,\xi)F(\xi,\xi) + U_n(\xi,\xi)F_\eta(\xi,\xi)]\,d\xi \\ &- \int_{1-\varepsilon}^1 (U_nF_\eta)(1-\varepsilon,\eta)\,d\eta + \int_{D_\varepsilon^{(1)}} (U_nF_{\xi\eta})(\xi,\eta)\,d\xi\,d\eta. \end{aligned} \tag{7.18}$$

>From (7.18) and (7.15) it follows that

$$\begin{aligned} K_1 \geq I_1 - I_2 = &-\int_0^{1-\varepsilon} [U_{n,\xi}(\xi,\xi)F(\xi,\xi) + U_n(\xi,\xi)F_\xi(\xi,\xi)]\,d\xi \\ &- \int_{1-\varepsilon}^1 (U_nF_\eta)(1-\varepsilon,\eta)\,d\eta + \int_{D_\varepsilon^{(1)}} U_n[F_{\xi\eta} - CF](\xi,\eta)\,d\xi\,d\eta. \end{aligned} \tag{7.19}$$

Because of (6.18), the last integral vanishes. Thus, using the boundary conditions for the functions U_n and F when $\eta = \xi$, we see that

$$\begin{aligned} K_1 \geq I_1 - I_2 = &-\textstyle\int_0^{1-\varepsilon}[U_{n,\xi}(\xi,\xi)F(\xi,\xi) + U_n(\xi,\xi)F_\xi(\xi,\xi)]\,d\xi \\ &- \int_{1-\varepsilon}^1 (U_nF_\eta)(1-\varepsilon,\eta)\,d\eta = -\frac{1}{2}(FU_n)(1-\varepsilon,1-\varepsilon) \\ &-\frac{1}{2}\int_0^{1-\varepsilon} \alpha(1-\xi)U_n(\xi,\xi)F(\xi,\xi)\,d\xi - \int_{1-\varepsilon}^1 (U_nF_\eta)(1-\varepsilon,\eta)\,d\eta. \end{aligned} \tag{7.20}$$

Since $\alpha(\xi), F_\eta \leq 0$ and $F, U_n, U_{n,\eta} \geq 0$, for $0 < \delta < \varepsilon < 1$ the relations

$$
\begin{aligned}
K_1 &\geq I_1 - I_2 \\
&\geq -\frac{1}{2}(U_nF)(1-\varepsilon,1-\varepsilon) + \int_{1-\varepsilon}^{1} U_n(1-\varepsilon,\eta)|F_\eta(1-\varepsilon,\eta)|\,d\eta \\
&\geq -\frac{1}{2}(U_nF)(1-\varepsilon,1-\varepsilon) + \int_{1-\delta}^{1} U_n(1-\varepsilon,\eta)|F_\eta(1-\varepsilon,\eta)|\,d\eta \\
&\geq -\frac{1}{2}(U_nF)(1-\varepsilon,1-\varepsilon) + \int_{1-\delta}^{1} U_n(1-\varepsilon,1-\delta)|F_\eta(1-\varepsilon,\eta)|\,d\eta \qquad (7.21)\\
&\geq -\frac{1}{2}(U_nF)(1-\varepsilon,1-\varepsilon) + (U_nF)(1-\varepsilon,1-\delta) \\
&\geq U_n(1-\varepsilon,1-\delta)\left[F(1-\varepsilon,1-\delta) - \frac{1}{2}F(1-\varepsilon,1-\varepsilon)\right] \\
&\geq \lambda(U_nF)(1-\varepsilon,1-\delta)
\end{aligned}
$$

hold, if the constant $\lambda > 0$ is such that

$$(1-\lambda)F(1-\varepsilon,1-\delta) \geq F(1-\varepsilon,1-\varepsilon). \tag{7.22}$$

Using the explicit formula (7.12) for the function $F(\xi,\eta)$, we see that the last inequality is equivalent to

$$(1-\lambda)\left(\frac{\delta}{\varepsilon+\delta}\right)^{n-\frac{1}{2}} \geq 2^{-n+\frac{1}{2}}, \tag{7.23}$$

which implies

$$0 < \lambda \leq 1 - \frac{1}{2}\left(\frac{\varepsilon+\delta}{2\delta}\right)^{n-\frac{1}{2}}. \tag{7.24}$$

A necessary condition, for (7.24) to be satisfied is

$$1 \leq \frac{\varepsilon}{\delta} < 2^{\frac{2n+1}{2n-1}} - 1. \tag{7.25}$$

In this concrete case, using (7.25), we can find an upper estimate for the generalized solution u_n To do this we consider the domain

$$D^\mu := \{(\xi,\eta) : 1-\eta \leq 1-\xi \leq \mu(1-\eta)\}, \tag{7.26}$$

where $1 \leq \mu < 2^{\frac{2n+1}{2n-1}} - 1$. Observe that

$$\inf_{D^\mu}\left\{1 - \frac{1}{2}\left(\frac{1-\xi+1-\eta}{2(1-\eta)}\right)^{n-\frac{1}{2}}\right\} = 1 - \frac{1}{2}\left(\frac{1+\mu}{2}\right)^{n-\frac{1}{2}} =: C_\mu > 0.$$

For $\lambda = C_\mu$ the inequalities (7.23) and (7.22) are satisfied and so, by (7.21), we see that

$$U(\xi,\eta) \leq 2^{-n+5/2}K_1C_\mu^{-1}\left(\frac{2-\xi-\eta}{(1-\xi)(1-\eta)}\right)^{n-\frac{1}{2}}, (\xi,\eta) \in D^\mu. \tag{7.27}$$

By (6.7) and (6.6), the inequality (7.27) transforms to

$$u_n^{(2)}(\varrho,t) \leq 4K_1C_\mu^{-1}\left(\frac{\varrho}{(\varrho+t)(\varrho-t)}\right)^{n-\frac{1}{2}}, \tag{7.28}$$

which is satisfied for

$$(\varrho,t)\in D_1^{\mu}:=\{0<\varrho-t\le\varrho+t\le\mu(\varrho-t)\}.$$

Finally, (7.28) implies

$$u_n^{(1)}(\varrho,t)\le 4K_1C_\mu^{-1}\varrho^{-1/2}\left(\frac{\varrho}{(\varrho+t)(\varrho-t)}\right)^{n-\frac{1}{2}}\quad\text{for }(\varrho,t)\in D_1^{\mu}, \tag{7.29}$$

which coincides with the estimate (7.4).

Note that $C_\mu=1/2$ on $\{t=0\}$ and so

$$u_n^{(1)}(\varrho,0)\le 8K_1\varrho^{-n},\qquad 0<\varrho<1, \tag{7.30}$$

which is the upper estimate in (7.5). The proof of Theorem 7.1 is complete. ■

REMARK 7.1. *Since in Theorems 6.1, 6.2 and 7.1 the conditions imposed upon lower terms of* (6.1) *are noninvariant with respect to substitution of the independent variables*

$$v(\varrho,\varphi,t)=u(\varrho,\varphi,t)\ \exp\lambda(\varrho,t), \tag{7.31}$$

for various functions $\lambda(\varrho,t)$ we can find a series of singular solutions of Problem P_α for different classes of equations of the form (6.1). *This procedure is interesting by itself and is demonstrated by the following simple*

Example 7.1. *Consider the special form of the equation* (6.1), *that is*

$$Lu\equiv\frac{1}{\varrho}(\varrho u_\varrho)_\varrho+\frac{1}{\varrho^2}u_{\varphi\varphi}-u_{tt}+2bu_t-b^2u=f,\ \text{in }\Omega_0,\quad b=const.\ne 0 \tag{7.32}$$

with the boundary conditions (6.2). *Obviously, the equation* (7.32) *does not satisfy the conditions of any of Theorems 6.1, 6.2, 7.1. Nevertheless, for $\alpha(\varrho)\in C^1([0,1])$ the equation* (7.32) *satisfies the conclusion of Theorem 6.1. This follows from* (7.31) *for $\lambda(\varrho,t)=-bt$. Indeed, in this case the equation* (7.32) *becomes* (7.1) *and the result follows by using Theorem 7.1.* ■

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SHARP CONDITIONS FOR OSCILLATION OF FUNCTIONAL EQUATIONS [1]

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ABSTRACT

This paper is concerned with the oscillation of second order linear functional equations of the form

$$x(g(t)) = P(t)x(t) + Q(t)x(g^2(t)),$$

where $P, Q, g : [t_0, \infty) \to R^+ = [0, \infty)$ are given real valued functions such that $g(t) \not\equiv t, \lim_{t\to\infty} g(t) = \infty$. It is proved here that when

$$0 \le m := \liminf_{t\to\infty} Q(t)P(g(t)) \le 1/4$$

all solutions of this equation oscillate if the condition

$$\limsup_{t\to\infty} Q(t)P(g(t)) > \left(\frac{1+\sqrt{1-4m}}{2}\right)^2 \qquad (*)$$

is satisfied. It should be emphasized that the condition $(*)$ can not be improved in some sense.

Keywords. Oscillation, nonoscillation, functional equation.

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1. INTRODUCTION

The oscillation theory of functional differential equations and difference equations with discrete arguments has been extensively developed during the last few decades. See, for example, [2,4-7,9,12] and the references cited therein. For the oscillatory behavior of functional equations which include difference equations with continuous arguments, the reader is referred to [1,3,8,10,11,13-15].

In 1992, Ladas, Pakula and Wang [8] considered the difference equation

$$x(t) + p_1 x(t - \tau_1) + p_2 x(t - \tau_2) = 0, \qquad p_1, p_2, \tau_1, \tau_2 \in R \tag{1.1}$$

and proved that every continuous solution of Eq. (1.1) oscillates if and only if the characteristic equation

$$1 + p_1 e^{-\lambda\tau_1} + p_2 e^{-\lambda\tau_2} = 0 \tag{1.2}$$

has no real roots. Observe that when $p_1, p_2 \in (0, \infty)$, every solution of Eq. (1.1) oscillates. Without loss of generality, it can be assumed that $\tau_1 > \tau_2 > 0$. But then $p_1 > 0$ is a necessary condition for all solutions of Eq. (1.1) to oscillate. On the basis of this observation it suffices to study the equation of the form

$$x(t) - px(t - \tau) + qx(t - \sigma) = 0, \tag{1.3}$$

where

$$p, q, \tau, \sigma \in (0, \infty) \quad \text{and} \quad \tau < \sigma.$$

In 1993 Domshlak [1], in 1996 Shen [11], in 1997 Zhang, Yan and Zhao [15] and in 1998 Zhang, Yan and Choi [14] studied such equations with variable coefficients, while in 1999, Yan and Zhang [13] considered a system of delay difference equations with constant coefficients.

In the above mentioned papers the equations under consideration are called difference equations with continuous argument (or continuous variable or continuous time) most likely because constant delays appear in these equations.

In 1994, Golda and Werbowski [3] studied the second order linear functional equation of the form

$$x(g(t)) = P(t)x(t) + Q(t)x(g^2(t)), \quad t \geq t_0, \tag{1.4}$$

where $P, Q, g : [t_0, \infty) \to R^+ = [0, \infty)$ are given real valued functions, $t_0 \geq 0$, x is an unknown real valued function, $g(t) \not\equiv t$ for $t \geq t_0$, $\lim_{t\to\infty} g(t) = \infty$, and g^m denotes the m-th iterate of the function g, i.e.,

$$g^0(t) = t, \;\; g^{i+1}(t) = g(g^i(t)), \;\; t \geq t_0, \;\; i = 0, 1, 2, \cdots.$$

They proved that all solutions of Eq. (1.4) oscillate if

$$\limsup_{t\to\infty} Q(t)P(g(t)) > 1 \tag{1.5}$$

or

$$\liminf_{t\to\infty} Q(t)P(g(t)) > \frac{1}{4}. \tag{1.6}$$

In the same paper they also improved condition (1.5) to

$$\limsup_{t\to\infty} \left\{ Q(t)P(g(t)) + \sum_{i=0}^{k} \prod_{j=0}^{i} Q(g^{j+1}(t))P(g^{j+2}(t)) \right\} > 1, \tag{1.7}$$

where $k \geq 0$ is some integer. In 1995, Nowakowska and Werbowski [10] extended condition (1.6) to higher order linear functional equations.

By a solution of (1.4) we mean a real valued function $x : [t_0, \infty) \to R$ such that $\sup\{|x(s)| : s \geq t^*\} > 0$ for any $t^* \geq t_0$ and x satisfies Eq. (1.4) on $[t_0, \infty)$. Such a solution is called oscillatory if there exists a sequence of points $\{t_n\}_1^\infty, t_n \in [t_0, \infty)$, such that $\lim_{n\to\infty} t_n = \infty$ and $x(t_n) \cdot x(t_{n+1}) \leq 0$ for $n = 1, 2, \cdots$. Otherwise it is called nonoscillatory.

Observe that in Eq. (1.4) $g(t) \not\equiv t$ but otherwise it may be $g(t) < t$ (retarded argument), $g(t) > t$ (advanced argument), or both $g(t) \leq t$ and $g(t) \geq t$ (mixed argument).

It should be noted that conditions (1.5) and (1.6) may be seen as the analogues of the well-known oscillation conditions (cf.[4-6,9])

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > 1,$$

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > \frac{1}{e},$$

for the functional differential equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \tag{1.8}$$

where $p \in C([t_0, \infty), R^+), \tau(t)$ is nondecreasing, $\tau(t) < t$ and $\lim_{t\to\infty} \tau(t) = \infty$.

As far as the lower bound 1/4 in the condition (1.6) is concerned, as it was pointed out in [3], it cannot be replaced by a smaller number. At this point it should be emphasized that (1.6) is a "sharp" condition in the sense that, when $P(t) \equiv p > 0, Q(t) \equiv q > 0$ and $g(t) = t - \tau$ $(\tau > 0)$, it reduces to

$$pq > 1/4 \tag{1.9}$$

which is a necessary and sufficient condition for the oscillation of

$$x(t - \tau) = px(t) + qx(t - 2\tau). \tag{1.10}$$

It is obvious that there is a gap between the conditions (1.5) and (1.6) when the limit $\lim_{t\to\infty} Q(t)P(g(t))$ does not exist. How to fill this gap is an interesting problem (cf.[2,5,6,9]). Here we should mention that condition (1.7) is an attempt in this direction. In fact, by setting

$$m := \liminf_{t\to\infty} Q(t)P(g(t)) \quad \text{and} \quad M := \limsup_{t\to\infty} Q(t)P(g(t)), \tag{1.11}$$

from condition (1.7) we can obtain that all solutions of (1.4) oscillate if $0 \le m \le 1/4$ and

$$M > \frac{1-2m}{1-m}. \tag{1.12}$$

Indeed, when $0 < m \le 1/4$ we can let $\varepsilon \in (0, m)$ be sufficiently small such that $Q(t)P(g(t)) \ge m - \varepsilon$ for t sufficiently large and by (1.12)

$$M > \frac{1-2(m-\varepsilon)}{1-(m-\varepsilon)} = 1 - \frac{m-\varepsilon}{1-(m-\varepsilon)}.$$

The last inequality, in view of the fact that $(m-\varepsilon)^k \to 0$ as $k \to \infty$, implies that for some integer $k > 1$

$$\begin{aligned} M &> 1 - \frac{(m-\varepsilon)[1-(m-\varepsilon)^{k+1}]}{1-(m-\varepsilon)} \\ &= 1-(m-\varepsilon)[1+(m-\varepsilon)+\cdots+(m-\varepsilon)^k], \end{aligned}$$

which leads to (1.7).

Our aim in this paper is to improve the methods previously used to show that the conditions (1.5) and (1.7) may be weaked to

$$M > \left(\frac{1+\sqrt{1-4m}}{2}\right)^2. \tag{1.13}$$

Our main results are as follows.

THEOREM. *Let m be as in (1.11). Assume that $0 \le m \le 1/4$ and that for some integer $k \ge 0$*

$$\limsup_{t\to\infty} \left\{ \overline{m}Q(t)P(g(t)) + \sum_{i=0}^{k} \overline{m}^i \prod_{j=0}^{i} Q(g^{j+1}(t))P(g^{j+2}(t)) \right\} > 1, \tag{1.14}$$

where

$$\overline{m} = \left(\frac{1+\sqrt{1-4m}}{2}\right)^{-1}.$$

Then all solutions of (1.4) oscillate.

COROLLARY. *Let m and M be as in* (1.11). *Assume that* $0 \leq m \leq 1/4$ *and that*

$$M > \left(\frac{1+\sqrt{1-4m}}{2}\right)^2. \tag{1.15}$$

Then all solutions of (1.4) *oscillate.*

Remark. It is to be noted that as $m \to 0$, the condition (1.14) reduces to the condition (1.7) and the conditions (1.15) and (1.12) reduce to the condition (1.5). However the improvement is clear as $0 < m \leq 1/4$ because

$$1 > \frac{1-2m}{1-m} > \left(\frac{1+\sqrt{1-4m}}{2}\right)^2.$$

It is interesting to observe that when $m \to 1/4$ condition (1.15) reduces to

$$M > 1/4.$$

We see therefore that our condition (1.15) essentially improves the previous known results. Moreover this condition is "sharp" in the sense that the lower bound 1/4 cannot be replaced by a smaller number.

Before the proofs of our main results, let us first give an example which illustrates the said improvement .

EXAMPLE. Consider the equation

$$x(t-2\sin^2 t) = x(t) + (\frac{1}{4} + \alpha\cos^2 t)x(t - 2\sin^2 t - 2\sin^2(t - 2\sin^2 t)), \tag{1.16}$$

where $g(t) = t - 2\sin^2 t$, $P(t) \equiv 1$, $Q(t) = \frac{1}{4} + \alpha\cos^2 t$, and $\alpha > 0$ is a constant. It is easy to see that

$$m = \liminf_{t\to\infty}(\frac{1}{4} + \alpha\cos^2 t) = \frac{1}{4},$$

$$M = \limsup_{t\to\infty}(\frac{1}{4} + \alpha\cos^2 t) = \frac{1}{4} + \alpha > \frac{1}{4}.$$

Thus, by Corollary all solutions of (1.16) oscillate. However, the condition (1.5) is satisfied only for $\alpha > 3/4$ and the condition (1.12) is satisfied only for $\alpha > 5/12$.

2. PROOFS

In what follows the numbers m and M are denoted by (1.11). Let

$$w_1(t) = \frac{x(g(t))}{x(g^2(t))} \cdot P(g(t)) \quad \text{and} \quad w_2(t) = \frac{x(g^2(t))}{x(g(t))} \cdot Q(t).$$

We begin with the preliminary analysis of asymptotic behavior of the functions $w_1(t)$ and $w_2(t)$ for a possible nonoscillatory solution $x(t)$ of (1.4) in the case when $0 \leq m \leq 1/4$. For convenience, we will assume that inequalities about values of functions are satisfied eventually for all large t.

LEMMA. *Assume that $0 \leq m \leq 1/4$ and that Eq. (1.4) has an eventually positive solution $x(t)$. Then*

$$\limsup_{t\to\infty} w_i(t) \leq \frac{1+\sqrt{1-4m}}{2}, \qquad i = 1, 2. \tag{2.1}$$

Proof. We first prove (2.1) for $i = 1$. From (1.4), we have $x(g(t)) \geq P(t)x(t)$. Since $g(t) \to \infty$ as $t \to \infty$, it follows that

$$x(g^2(t)) \geq P(g(t))x(g(t)). \tag{2.2}$$

Thus $\limsup_{t\to\infty} w_1(t) \leq 1$, which implies (2.1) for $i = 1$ and $m = 0$. We now consider the case when $0 < m \leq 1/4$. For any $\eta \in (0, m)$, we have for sufficiently large t

$$Q(t)P(g(t)) \geq m - \eta. \tag{2.3}$$

From (2.2), we have

$$\frac{x(g(t))}{x(g^2(t))} P(g(t)) \leq 1 := d_1,$$

or

$$\frac{x(g^2(t))}{x(g(t))} \geq d_1^{-1} P(g(t)). \tag{2.4}$$

Dividing both sides of (1.4) by $x(g(t))$ and using (2.4), we have

$$1 \geq P(t)\frac{x(t)}{x(g(t))} + d_1^{-1} Q(t)P(g(t)). \tag{2.5}$$

Substituting (2.3) into (2.5), we have

$$1 \geq P(t)\frac{x(t)}{x(g(t))} + d_1^{-1}(m - \eta),$$

which yields

$$w_1(t) \leq \frac{d_1 - (m-\eta)}{d_1} := d_2.$$

Following this iterative procedure, we obtain

$$w_1(t) \leq \frac{d_n - (m-\eta)}{d_n} := d_{n+1}, \qquad n = 1, 2, \cdots. \tag{2.6}$$

It is not difficult to see that d_n is decreasing and hence the limit $\lim_{n\to\infty} d_n = d$ exists and satisfies $d^2 - d + (m - \eta) = 0$. Therefore, we have

$$\limsup_{t\to\infty} w_1(t) \leq \frac{1 + \sqrt{1 - 4(m-\eta)}}{2}.$$

Letting $\eta \to 0$, we obtain (2.1) for $i = 1$.

Next we prove (2.1) for $i = 2$. From (1.4), we have

$$x(g(t)) \geq Q(t)x(g^2(t)), \tag{2.7}$$

$$x(g^2(t)) = P(g(t))x(g(t)) + Q(g(t))x(g^3(t)). \tag{2.8}$$

From (2.7), it follows that

$$w_2(t) = \frac{x(g^2(t))}{x(g(t))}Q(t) \leq 1 := d_1.$$

Thus (2.1) holds for $i = 2$ and $m = 0$. We now assume that $0 < m \leq 1/4$. Then, for any $\eta \in (0, m)$ we have (2.3) holds. Thus, by (2.8) we have

$$\begin{aligned} 1 &= P(g(t))\frac{x(g(t))}{x(g^2(t))} + Q(g(t))\frac{x(g^3(t))}{x(g^2(t))} \\ &\geq P(g(t))\frac{Q(t)}{d_1} + Q(g(t))\frac{x(g^3(t))}{x(g^2(t))} \\ &\geq \frac{m-\eta}{d_1} + Q(g(t))\frac{x(g^3(t))}{x(g^2(t))}, \end{aligned}$$

which implies that for sufficiently large t

$$w_2(t) \leq \frac{d_1 - (m-\eta)}{d_1} := d_2.$$

Following this iterative procedure, we obtain

$$w_2(t) \leq \frac{d_n - (m-\eta)}{d_n} := d_{n+1}, \qquad n = 1, 2, \cdots.$$

Now the conclusion follows from the above inequalities by similar arguments as in the proof of (2.1) for $i = 1$. The proof is complete.

Proof of Theorem. Suppose to the contrary that Eq. (1.4) has an eventually positive solution $x(t)$. By Lemma, for any sufficiently small $\eta > 0$, the following inequalities hold for t sufficiently large

$$x(g(t)) \geq \overline{m}_\eta P(t)x(t), \tag{2.9}$$

$$x(g(t)) \geq \overline{m}_\eta Q(t)x(g^2(t)), \tag{2.10}$$

where

$$\overline{m}_\eta = \left(\frac{1+\sqrt{1-4m}}{2} + \eta\right)^{-1}.$$

From (2.9), by induction, we have the iterative formula

$$x(g^i(t)) \geq \left(\overline{m}_\eta^i \prod_{j=0}^{i-1} P(g^j(t))\right) x(t), \quad i = 1, 2, \cdots. \tag{2.11}$$

By (1.4), we have (2.8) holds. From (2.8), by induction, we have

$$x(g^{i+1}(t)) = P(g^i(t))x(g^i(t)) + Q(g^i(t))x(g^{i+2}(t)), \quad i = 2, 3, \cdots. \tag{2.12}$$

From (2.8) and (2.12), as in the proof of Theorem 2 in [3], we have for $k \geq 0$

$$\begin{aligned} x(g^2(t)) &= P(g(t))x(g(t)) + \sum_{i=0}^{k} P(g^{i+2}(t))x(g^{i+2}(t)) \prod_{j=0}^{i} Q(g^{j+1}(t)) \\ &\quad + x(g^{k+4}(t)) \prod_{j=0}^{k+1} Q(g^{j+1}(t)), \end{aligned}$$

which, in view of (2.10) and (2.11), gives

$$\begin{aligned} x(g^2(t)) &\geq \overline{m}_\eta Q(t)P(g(t))x(g^2(t)) \\ &\quad + x(g^2(t)) \sum_{i=0}^{k} P(g^{i+2}(t))\overline{m}_\eta^i \prod_{j=0}^{i-1} P(g^{j+2}(t)) \prod_{j=0}^{i} Q(g^{j+1}(t)) \\ &= \overline{m}_\eta Q(t)P(g(t))x(g^2(t)) \\ &\quad + x(g^2(t)) \sum_{i=0}^{k} \overline{m}_\eta^i \prod_{j=0}^{i} Q(g^{j+1}(t))P(g^{j+2}(t)). \end{aligned}$$

Dividing both sides of the above inequality by $x(g^2(t))$ and taking the limit as $t \to \infty$, we obtain

$$1 \geq \limsup_{t\to\infty} \left\{ \overline{m}_\eta Q(t)P(g(t)) + \sum_{i=0}^{k} \overline{m}_\eta^i \prod_{j=0}^{i} Q(g^{j+1}(t))P(g^{j+2}(t)) \right\}.$$

Letting $\eta \to 0$, we have

$$1 \geq \limsup_{t\to\infty} \left\{ \overline{m}Q(t)P(g(t)) + \sum_{i=0}^{k} \overline{m}^i \prod_{j=0}^{i} Q(g^{j+1}(t))P(g^{j+2}(t)) \right\},$$

which constradicts (1.14). The proof is complete.

Proof of Corollary. The conclusion clearly holds when $m = 0$. When $0 < m \leq 1/4$, it suffices to prove that (1.15) implies (1.14). Indeed, by (1.15) and noting that

$$\frac{1+\sqrt{1-4m}}{2} = 1 - \frac{m}{1-\overline{m}m},$$

there exists an $\varepsilon \in (0, m)$ such that $Q(t)P(g(t)) \geq m - \varepsilon$ for t sufficiently large and

$$\overline{m}M > 1 - \frac{m-\varepsilon}{1-\overline{m}(m-\varepsilon)}.$$

The last inequality, in view of the fact that $[\overline{m}(m-\varepsilon)]^k \to 0$ as $k \to \infty$, implies that for some integer $k > 1$

$$\begin{aligned} \overline{m}M &> 1 - \frac{(m-\varepsilon)\{1-[\overline{m}(m-\varepsilon)]^{k+1}\}}{1-\overline{m}(m-\varepsilon)} \\ &= 1 - [(m-\varepsilon) + \overline{m}(m-\varepsilon)^2 + \cdots + \overline{m}^k(m-\varepsilon)^{k+1}], \end{aligned}$$

which leads to (1.14). Thus, by Theorem, all solutions of (1.4) oscillate. The proof is complete.

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A note on defining information in random censoring

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Summary

In this paper we show that the two ways of defining information in random censoring, the Hollander, Proschan and Sconing approach and the classical or likelihood approach, lead to the same measure of information, if censoring is non-informative.

Keywords: Fisher information; φ-Divergence; Non-informative censoring.

1. Introduction

Information measures can be classified as parametric or Fisher-type and nonparametric or divergence-type [cf. Ferentinos and Papaioannou (1981), Papaioannou (1985), and references therein]. The main representative of the first type is Fisher's measure which is defined for parametric families of distributions, while for the second type, Csiszar's φ-divergence or the Kullback-Leibler measure of information are the main representatives [cf. Csiszar (1963), Ali and Silvey (1966), and Kullback and Leibler (1951)].

In random censoring, where X is the variable of interest (e.g. failure times) and Y the censoring variable, both parametric and nonparametric measures of information can be defined in two ways. The first way is to condition first on Y and then average Y out. The second is to evaluate information based on the likelihood of a single observation, this observation being either the true value X or the value of the censoring Y, if X is censored.

The first approach is known as the Hollander, Proschan and Sconing way of defining measures of information for censored data. It is based primarily on entropy [cf. Hollander *et al.* (1987) and (1990), Baxter (1989), and Stute (1992)]. It provides a formal definition of measures of information and has some nice properties which Hollander *et al.* (1987) call the "acid test", and which are pertinent to censored data. The second is known as the classical or the likelihood approach [cf. Tsairidis *et al.* (1996)] and has formed the basis for developing results in statistical information theory. The two approaches do not lead, in general, to the same value for the measure.

The merits of these two approaches in the context of random censoring have been compared in Tsairidis *et al.* (1996). On the basis of various fundamental properties of the measures, the likelihood approach was preferred. In a given situation, however, and with respect to maximum likelihood estimation in particular, there still exists a dilemma as to which approach to use. Stute (1992), for example, studies the strong consistency of the maximum likelihood estimator in random censoring and for his measure of infomation follows the Hollander, Proschan and Sconing approach. Miller (1981), however, in his classic book uses the likelihood approach. The same is true for Abdushukurov and Kim (1987) who study Cramer-Rao and Bhattacharyya bounds for randomly censored observations. Thus, it is of interest to explore the cases where the two approaches lead to the same value, in view also of the computational effort involved for the evaluation of each measure due to the appearance of c.d.f's in their definitions. The purpose of this note is to clarify further the issues regarding the definition of information in random censoring, and primarily to show that, for both Fisher-type and divergence-type measures of information, the two ways of defining information in random censoring lead to the same measures of information, if censoring is non-informative.

Censoring is called non-informative if the likelihood function can be decomposed into the product of two components: one that involves only parameters for the distribution of the variable X of interest and the other only parameters for the distribution of variable Y or no parameters at all. Parameters in the latter component can be disregarded in making inferences about the distribution of failure times [cf. Kalbfleisch and Prentice (1980), p. 121, Lawless (1982), p. 42, and Wu and Carroll (1988)]. There are other aspects of non-informativeness such as conditional independence between censoring and failure times for a given set of covariates [cf. Slud *et al.* (1988)] or the constant-sum condition [cf. Kalbfleisch and Mackay (1979), and Williams and Lagakos (1977)]. Here we shall assume that Y has a fixed, possibly known, distribution or that it does not depend on unknown parameters. The mode is non-Bayesian.

The statistical and notational setup will be as follows: X will be the variable of interest, for example, the time taken for an event to occur, and Y the censoring variable, independent of X. In right random censoring we observe $Z = \min(X, Y)$ and $\delta = I_{(X \leq Y)}$. If $\delta = 1$ we observe $Z = X$, an uncensored observation. If $\delta = 0$ we observe $Z = Y$, a censored observation. F will be the c.d.f. of X and G the c.d.f. of Y, and $\overline{F}$ and $\overline{G}$ the corresponding survival functions. f and g will be the corresponding p.d.f's.

For Fisher-type measures of information we shall assume that the distribution of X belongs to a parametric family of distributions $\mathcal{M} = \{P_\theta, \theta \in \Theta\}$ with p.d.f. $f(x;\theta) = \frac{dP_\theta}{d\mu}$, relative to a dominating measure μ on the real space $\mathcal{X}$, satisfying some appropriate regularity conditions [cf. Papaioannou (1985)]. $\mathcal{X}$ is nonnegative and the parameter space is an open subset of the Euclidean space R^k. The distribution of Y does not depend on θ. We denote by $p\left((z,\delta);\theta\right)$ the p.d.f. of (Z,δ). The distribution of (Z,δ) is given by

$$p\left((z,\delta);\theta\right) = \left[f(z;\theta)\overline{G}(z)\right]^{\delta}\left[g(z)\overline{F}(z;\theta)\right]^{1-\delta}, \ \delta = 0,1. \tag{1.1}$$

For divergence-type measures of information we consider two probability measures P_1 and P_2 on the measurable space $(\mathcal{X},\mathcal{T})$, $\mathcal{T}$ a σ-algebra associated with $\mathcal{X}$. Let $f_i(x) = \frac{dP_i}{d\mu}$ and $F_i(x)$, $i = 1,2$, be the p.d.f and c.d.f of X, respectively, where μ is a dominating finite or σ-finite measure on $\mathcal{X}$. For the censoring variable Y we shall assume a p.d.f. $g(y)$ and a c.d.f. $G(y)$ as before, independent of θ. The distribution of Y need not belong to $\mathcal{M}$. For the φ-divergence or Csiszar's measure of information we shall also assume that φ is a convex function satisfying appropriate regularity conditions [cf. Csiszar (1963)]. Again the distribution of (Z,δ), for each (f_i, F_i) is given by (1.1) without θ.

We shall use the notation $I_{(X,Y)}$ and $I_{(Z,\delta)}$ to denote the measures of information defined by Hollander, Proschan and Sconing, and the likelihood approach, respectively.

The result of this note is helpful particularly when one considers the asymptotic distribution of maximum likelihood estimates and wonders whether to use the one or the other approach in defining Fisher's information for the asymptotic variance, each having certain computational advantages.

2. Main results

2.1 Fisher-type measures of information

In view of (1.1), Fisher's measure of information about θ contained in the censored data (Z,δ) and based on the likelihood of a single observation is given by

$$\begin{aligned} I^F_{(Z,\delta)}(\theta) &= \left\| \sum_{\delta=0,1}\int_{-\infty}^{\infty} p\left((z,\delta);\theta\right)\left(\tfrac{\partial}{\partial\theta_i}\log p\left((z,\delta);\theta\right)\tfrac{\partial}{\partial\theta_j}\log p\left((z,\delta);\theta\right)\right)dz \right\|_{k\times k} \\ &= \left\| \int_{-\infty}^{\infty} \tfrac{f_{(i)}(z;\theta)f_{(j)}(z;\theta)}{f(z;\theta)}\overline{G}(z)dz + \int_{-\infty}^{\infty} \tfrac{\overline{F}_{(i)}(z;\theta)\overline{F}_{(j)}(z;\theta)}{\overline{F}(z;\theta)}g(z)dz \right\|_{k\times k}, \end{aligned} \tag{2.1}$$

($\|\cdot\|_{k\times k}$ denotes a $k\times k$ matrix), where

$$f_{(i)}(z;\theta) = \tfrac{\partial f(z;\theta)}{\partial\theta_i}, \qquad \overline{F}_{(i)}(z;\theta) = \tfrac{\partial\overline{F}(z;\theta)}{\partial\theta_i}, \qquad i,j = 1,2,\ldots,k.$$

If one uses the Hollander, Proschan and Sconing approach, Fisher's amount of information about θ contained in the censored experiment (X, Y) must be defined by

$$I^F_{(X,Y)}(\theta) = \left\| \int_{-\infty}^{\infty} g(y) \left[\int_{-\infty}^{y} \frac{f_{(i)}(x;\theta) f_{(j)}(x;\theta)}{f(x;\theta)} dx + \frac{F_{(i)}(y;\theta) F_{(j)}(y;\theta)}{\overline{F}(y;\theta)} \right] dy \right\|_{k \times k}, \tag{2.2}$$

[cf. Tsairidis *et al.* (1996) and (2001)]. One advantage of the Hollander, Proschan and Sconing definition is that it does not require knowledge in closed form of G (and $\overline{G}$) which in many cases is not available.

If censoring is informative, i.e., the distribution of Y depends on θ, Y provides information about θ, and this should be taken into account in defining measures of information. Thus, it is not appropriate to use formally the Hollander, Proschan and Sconing approach to define Fisher-type measures of information, as in (2.2), since we should also consider the score based on $g(y;\theta)$.

Theorem 2.1 In non-informative right random censoring we have

$$I^F_{(Z,\delta)}(\theta) = I^F_{(X,Y)}(\theta), \ \theta \in \Theta.$$

Proof. Only the case where θ is univariate will be treated. The k-variate case is straightforward. To prove the equivalence of (2.1) and (2.2), it is enough to show that

$$\int_{-\infty}^{\infty} \frac{(f_\theta(x;\theta))^2}{f(x;\theta)} \overline{G}(x) dx = \int_{-\infty}^{\infty} g(y) \left[\int_{-\infty}^{y} \frac{(f_\theta(x;\theta))^2}{f(x;\theta)} dx \right] dy,$$

where differentiation with respect to θ is denoted by $f_\theta(x;\theta) = \frac{\partial f(x;\theta)}{\partial \theta}$. Note also that $\overline{F}_\theta(\cdot;\theta) = -F_\theta(\cdot;\theta)$. We have

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{(f_\theta(x;\theta))^2}{f(x;\theta)} \overline{G}(x) dx &= \int_{-\infty}^{\infty} \frac{(f_\theta(x;\theta))^2}{f(x;\theta)} dx - \int_{-\infty}^{\infty} \frac{(f_\theta(x;\theta))^2}{f(x;\theta)} G(x) dx \\
&= I^F_X(\theta) - \int_{-\infty}^{\infty} \frac{(f_\theta(x;\theta))^2}{f(x;\theta)} \left[\int_{-\infty}^{x} g(y) dy \right] dx \\
&= I^F_X(\theta) - \int_{-\infty}^{\infty} g(y) \left[\int_{y}^{\infty} \frac{(f_\theta(x;\theta))^2}{f(x;\theta)} dx \right] dy \\
&= I^F_X(\theta) - \int_{-\infty}^{\infty} g(y) \left[I^F_X(\theta) - \int_{-\infty}^{y} \frac{(f_\theta(x;\theta))^2}{f(x;\theta)} dx \right] dy \\
&= \int_{-\infty}^{\infty} g(y) \left[\int_{-\infty}^{y} \frac{(f_\theta(x;\theta))^2}{f(x;\theta)} dx \right] dy.
\end{aligned}$$

Thus

$$I^F_{(Z,\delta)}(\theta) = I^F_{(X,Y)}(\theta).$$

Remark 2.1 A weakness of the Hollander, Proschan and Sconing approach is that it does not lead to a natural definition of the observed Fisher information [cf. Efron and Hinkley (1978)] which has certain advantages over the expected Fisher information. For measures derived from the Fisher information matrix which have been proposed as simpler measures of information, such as the trace, the determinant, the eigenvalues etc or even Vajda's measure of information, the result of Theorem 1 remains true [cf. Ferentinos and Papaioannou (1981)].

2.2 Divergence-type measures of information

Csiszar's φ-divergence between f_1 and f_2 based on the randomly censored experiment at hand and the likelihood, is the φ-divergence between $p_1(z,\delta)$ and $p_2(z,\delta)$ which are generated from f_1 and f_2 via (1.1). Then it is defined as

$$\begin{aligned} I^C_{(Z,\delta)}(f_1,f_2) &= \sum_{\delta=0,1} \int_0^\infty p_2(z,\delta)\varphi\left(\tfrac{p_1(z,\delta)}{p_2(z,\delta)}\right) dz \\ &= \int_0^\infty f_2(z)\overline{G}(z)\varphi\left(\tfrac{f_1(z)}{f_2(z)}\right) dz + \int_0^\infty g(z)\overline{F}_2(z)\varphi\left(\tfrac{\overline{F}_1(z)}{\overline{F}_2(z)}\right) dz. \end{aligned}$$

But if one uses the Hollander, Proschan and Sconing approach, Csiszar's φ-divergence between f_1 and f_2 based on the censored experiment (X,Y) must be defined as follows [cf. Tsairidis *et al.* (1996) and (2001)]

$$I^C_{(X,Y)}(f_1,f_2) = \int_{-\infty}^{\infty} g(y)\left[\int_{-\infty}^{y} f_2(x)\varphi\left(\frac{f_1(x)}{f_2(x)}\right) dx + \overline{F}_2(y)\varphi\left(\frac{\overline{F}_1(y)}{\overline{F}_2(y)}\right)\right] dy. \tag{2.3}$$

Again the evaluation of $I^C_{(X,Y)}(f_1,f_2)$ does not require knowledge of $\overline{G}$.

Note however that in the informative case, Csiszar's measure of information may not be formally defined via (2.3), since we could also have two p.d.f's for Y. For this divergence-type measure of information we then have:

Theorem 2.2 In non-informative right random censoring we have

$$I^C_{(Z,\delta)}(f_1,f_2) = I^C_{(X,Y)}(f_1,f_2).$$

Proof. It is enough to show that

$$\int_{-\infty}^{\infty} f_2(x)\overline{G}(x)\varphi\left(\frac{f_1(x)}{f_2(x)}\right) dx = \int_{-\infty}^{\infty} g(y)\left[\int_{-\infty}^{y} f_2(x)\varphi\left(\frac{f_1(x)}{f_2(x)}\right) dx\right] dy.$$

We have

$$\begin{aligned}
\int_{-\infty}^{\infty} f_2(x)\overline{G}(x)\varphi\left(\tfrac{f_1(x)}{f_2(x)}\right)dx &= \int_{-\infty}^{\infty} f_2(x)\varphi\left(\tfrac{f_1(x)}{f_2(x)}\right)dx - \int_{-\infty}^{\infty} f_2(x)G(x)\varphi\left(\tfrac{f_1(x)}{f_2(x)}\right)dx \\
&= I_X^C(f_1,f_2) - \int_{-\infty}^{\infty}\left[\int_{-\infty}^{x} g(y)dy\right] f_2(x)\varphi\left(\tfrac{f_1(x)}{f_2(x)}\right)dx \\
&= I_X^C(f_1,f_2) - \int_{-\infty}^{\infty} g(y)\left[\int_{y}^{\infty} f_2(x)\varphi\left(\tfrac{f_1(x)}{f_2(x)}\right)dx\right]dy \\
&= I_X^C(f_1,f_2) - \int_{-\infty}^{\infty} g(y)I_X^C(f_1,f_2)dy \\
&\qquad + \int_{-\infty}^{\infty} g(y)\left[\int_{-\infty}^{y} f_2(x)\varphi\left(\tfrac{f_1(x)}{f_2(x)}\right)dx\right]dy \\
&= \int_{-\infty}^{\infty} g(y)\left[\int_{-\infty}^{y} f_2(x)\varphi\left(\tfrac{f_1(x)}{f_2(x)}\right)dx\right]dy.
\end{aligned}$$

So

$$I_{(Z,\delta)}^C(f_1,f_2) = I_{(X,Y)}^C(f_1,f_2).$$

3. Conclusion

The results of this note, applicable in non-informative random censoring, are helpful when one considers the asymptotic distribution of maximum likelihood estimators for the parameters of the model and wonders whether to use the one or the other approach in defining Fisher's information for the asymptotic variance. Each approach has certain computational advantages, but both lead to the same value. If the c.d.f. G of the censoring variable is not available in closed form, a case not rare in practice, it may be easier to use (2.2) to compute Fisher's information than (2.1), which requires knowledge of both c.d.f's F and G. The same is true for φ-divergence.

The reader may note that in informative random censoring, i.e., when the distribution of the censoring variable Y depends on the same parameter θ of the model or another parameter η, the equivalence shown above is not true [cf. Tsairidis *et al.* (1996)].

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Four inventory models for deteriorating items with time varying demand and partial backlogging: A cost comparison

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Abstract. In this paper we study four inventory models under the following assumptions. Planning horizon is finite and demand varies over time in a deterministic way. The models allow for deterioration of items over time and partial backlogging of unsatisfied demand. For each of the models we obtain the optimal replenishment policy. We, then make a comparison of the four models, which ranks them according to their optimal costs.

Keywords: Inventory, deterioration, shortages, partial backlogging, comparison.

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1. Introduction

Inventory models with deteriorating items have been extensively studied, and up to now a large number of research articles have appeared in the literature. In a recent paper, Teng et al.[1] studied four inventory models with deteriorating items and complete backlogging of unsatisfied demand. For these models they gave the optimal replenishment policies and made a comparison between the optimal relevant costs. The fluctuation of the inventory level at anyone of the four models is given in figures 1, 2, 3, 4. The traditional model (model I) starts with replenishment and ends with no shortages. It has been used by Deb and Chaudhuri[2], Dave[3]. Model II starts with a replenishment and ends without shortages. It has been studied by Sachan[4], Benkherouf and Mahmoud[5]. Model III starts and ends with no shortages. It has been used by Goyal et al.[6], Hariga[7], Chakrabarti and Chaudhuri[8]. Model IV starts with shortages and ends with shortages. It has been introduced by Teng et al.[9] The first three models are repeatable. For repeatability of model IV we referred to Teng et al.[9] Of course here we examine all models in a finite prescribe horizon.

All above-mentioned studies assumed complete backlogging of unsatisfied demand. Here we extend the above results to cover the case where unsatisfied demand is partially backlogged with an exponentially time dependent rate. More explicitly we suppose that the rate of backlogged demand increases exponentially as the waiting time to the next replenishment decreases. This assumption seems quite reasonable since as the waiting time to the next replenishment decreases more and more customers do accept to wait and get their goods as soon as the stock of the next replenishment enters the system. Moreover it adds flexibility to the system as we can vary the rate of backlogging. Researches that used the idea of partial backlogging are these of Wee[10, 11], Wee and Mercan[12], Chang and Dye[13] and others. The most of them consider that only a constant fraction of the demand during the stockout period is satisfied the remaining is lost.

The paper is organized as follows. In section 2 we give the assumptions of the models and the notation used. We continue in section 3 with the mathematical formulation of the models. In section 4 we present their optimal replenishment policies. A comparison between the total cost functions of the proposed four models is made in section 5. Finally, section 6 contains some concluding remarks.

2. Assumptions and notations

The inventory model is a continuous review model developed under the following assumptions.

Notation:

n : the number of replenishment cycles during the planning horizon.

t_I : time at which the *i*th replenishment is made i=1,…,n (reorder points).

s_i : time at which shortage starts in the *i*th cycle (t_i, t_{i+1}) i=1,…,n-1 or n (shortage points).

C : the replenishment cost per order.

C_1 : holding cost per unit of stock carried per unit time.

C_2 : shortage cost per unit of shortage per unit time.

C_3 : deterioration cost per unit of deteriorated items.

C_4 : opportunity cost due to the lost sales per unit of lost sales items.

CI_I: the amount of inventory carried during the *i*th cycle.

DI_i: the amount of deteriorated items during the *i*th cycle.

SI_i : the amount of units in shortages during the *i*th cycle.

LI_i : the amount of lost sales units during the *i*th cycle.

I(t): the inventory level at time t.

Assumptions

1. The planning horizon of the system is finite and is taken as H time units.
2. Replenishment is instantaneous (replenishment rate is infinite).
3. The lead-time is zero.
4. The on-hand inventory deteriorates at a constant rate θ ($0<\theta<1$) per time unit. The deteriorated items are withdrawn immediately from the warehouse and there is no provision for repair or replacement.
5. The rate of demand at any time, $t \in [0, H]$, is a continuous, logconcave function f(t), of t, with $f'(t) \neq 0\ \forall$ t.
6. The system allows for shortages.
7. Unsatisfied demand is backlogged at a rate $\exp(-\alpha x)$, where x is the time up to the next replenishment and α parameter with, $0<\alpha<\frac{C_2}{C_2H-C_4}$, $C_2H-C_4>0$.

1. The mathematical formulation of the model

The depletion of inventory during the interval $[t_i, s_i]$, of the *i*th replenishment cycle, is due to the joint effect of the demand and the deterioration of items. Hence the differential equation, which describes the variation of inventory level, I(t), with respect to time, t, is:

$$\frac{dI(t)}{dt} = -\theta I(t) - f(t), \quad t_i \le t \le s_i \tag{1}$$

with boundary condition $I(s_i)=0$.

The solution of this equation is

$$I(t) = e^{-\theta t}\int_t^{s_i} e^{\theta u} f(u)du, \quad t_i \le t \le s_i \tag{2}$$

Using (2) we obtain the amount of inventory carried during the *i*th cycle

$$CI_i = \frac{1}{\theta}\int_{t_i}^{s_i}(e^{\theta(t-t_i)} - 1)f(t)dt \tag{3}$$

The amount of deteriorated items during the *i*th cycle is now given by

$$DI_i = \theta CI_i = \int_{t_i}^{s_i}(e^{\theta(t-t_i)} - 1)f(t)dt \tag{4}$$

The variation of the inventory level, I(t), with respect to time, t, during interval $[s_i, t_{i+1}]$ can be described by the differential equation

$$\frac{dI(t)}{dt} = -e^{-\alpha(t_{i+1}-t)}f(t), \quad s_i \le t \le t_{i+1} \tag{5}$$

with boundary condition $I(s_i)=0$

The solution to (5) is

$$I(t) = -\int_{s_i}^{t} e^{-\alpha(t_{i+1}-u)}f(u)du, \quad s_i \le t \le t_{i+1} \tag{6}$$

From (6) we obtain the amount of shortage during the *i*th cycle

$$SI_i = \int_{s_i}^{t_{i+1}}\int_{s_i}^{t} e^{-\alpha(t_{i+1}-u)}f(u)dudt = \int_{s_i}^{t_{i+1}} e^{-\alpha(t_{i+1}-u)}(t_{i+1}-u)f(u)du \tag{7}$$

The amount of lost sales during the *i*th cycle is

$$LI_i = \int_{s_i}^{t_{i+1}} (1-e^{-\alpha(t_{i+1}-t)})f(t)dt \tag{8}$$

Now we have all necessary quantities to formulate the total inventory cost function, for the four models for any policy with n replenishments. The cost functions are made up as the sum of the ordering cost, holding cost, deterioration cost, shortage cost and opportunity cost due to the lost sales:

$$TC_1(n,s_i,t_i)=nC+C_1\sum_{i=1}^{n} CI_i + C_2\sum_{i=1}^{n-1} SI_i + C_3\sum_{i=1}^{n} DI_i + C_4\sum_{i=1}^{n-1} LI_i$$

$$=nC+\frac{C_1+C_3\theta}{\theta}\sum_{i=1}^{n}\int_{t_i}^{s_i}(e^{\theta(t-t_i)}-1)f(t)dt+\sum_{i=1}^{n-1}\int_{s_i}^{t_{i+1}}(C_4-C_4e^{-\alpha(t_{i+1}-t)}+C_2e^{-\alpha(t_{i+1}-t)}(t_{i+1}-t))f(t)dt \tag{9}$$

with $t_1=0$, $s_n=H$

$$TC_2(n,s_i,t_i)=nC+C_1\sum_{i=1}^{n} CI_i + C_2\sum_{i=1}^{n} SI_i + C_3\sum_{i=1}^{n} DI_i + C_4\sum_{i=1}^{n} LI_i$$

$$=nC+\frac{C_1+C_3\theta}{\theta}\sum_{i=1}^{n}\int_{t_i}^{s_i}(e^{\theta(t-t_i)}-1)f(t)dt+\sum_{i=1}^{n}\int_{s_i}^{t_{i+1}}(C_4-C_4e^{-\alpha(t_{i+1}-t)}+C_2e^{-\alpha(t_{i+1}-t)}(t_{i+1}-t)f(t)dt \tag{10}$$

with $t_1=0$, $t_{n+1}=H$.

$$TC_3(n,s_i,t_i)=nC+C_1\sum_{i=1}^{n} CI_i + C_2\sum_{i=0}^{n-1} SI_i + C_3\sum_{i=1}^{n} DI_i + C_4\sum_{i=0}^{n-1} LI_i$$

$$=nC+\frac{C_1+C_3\theta}{\theta}\sum_{i=1}^{n}\int_{t_i}^{s_i}(e^{\theta(t-t_i)}-1)f(t)dt+\sum_{i=0}^{n-1}\int_{s_i}^{t_{i+1}}(C_4-C_4e^{-\alpha(t_{i+1}-t)}+C_2e^{-\alpha(t_{i+1}-t)}(t_{i+1}-t))f(t)dt \tag{11}$$

with $s_0=0$, $s_n=H$.

$$TC_4(n,s_i,t_i)=nC+C_1\sum_{i=1}^{n} CI_i + C_2\sum_{i=0}^{n} SI_i + C_3\sum_{i=1}^{n} DI_i + C_4\sum_{i=0}^{n} LI_i$$

$$=nC+\frac{C_1+C_3\theta}{\theta}\sum_{i=1}^{n}\int_{t_i}^{s_i}(e^{\theta(t-t_i)}-1)f(t)dt+\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}}(C_4-C_4e^{-\alpha(t_{i+1}-t)}+C_2e^{-\alpha(t_{i+1}-t)}(t_{i+1}-t))f(t)dt \tag{12}$$

with $s_0=0$, $t_{n+1}=H$.

4. The optimal replenishment policy

In this section we shall present all the results which will insure the existence of the optimal t_i, s_i values, for any policy with n replenishments and we shall give the algorithm which can be used to find them. Then, we shall prove the convexity of the total cost function, $TC(n, s_i, t_i)$, w.r.t. n, a fact which guarantees the existence of the over all optimal policy.

The continuity of f(t) guarantees that $TC_j(n, s_i, t_i)$ j=1,2,3,4 is a continuous function of s_i, t_i and its first and second order partial derivatives exist. So, taking first order derivatives of $TC_j(n, s_i, t_i)$ w. r. t t_i, s_i, and equating them to zero we obtain

Model I

$$(C_1 + C_3\theta)\int_{t_i}^{s_i} e^{\theta(t-t_i)} f(t)dt = \int_{s_{i-1}}^{t_i} [\alpha C_4 + C_2 - \alpha C_2(t_i - t)] e^{-\alpha(t_i - t)} f(t)dt\ , \quad i=2,\dots,n \tag{13}$$

$$\frac{C_1 + C_3\theta}{\theta}[e^{\theta(s_i - t_i)} - 1] = C_2(t_{i+1} - s_i)e^{-\alpha(t_{i+1} - s_i)} + C_4(1 - e^{-\alpha(t_{i+1} - s_i)}),\quad i=1, 2,\dots,n-1 \tag{14}$$

Model II

$$(C_1 + C_3\theta)\int_{t_i}^{s_i} e^{\theta(t-t_i)} f(t)dt = \int_{s_{i-1}}^{t_i} [\alpha C_4 + C_2 - \alpha C_2(t_i - t)] e^{-\alpha(t_i - t)} f(t)dt\ , \quad i=2,\dots,n \tag{15}$$

$$\frac{C_1 + C_3\theta}{\theta}[e^{\theta(s_i - t_i)} - 1] = C_2(t_{i+1} - s_i)e^{-\alpha(t_{i+1} - s_i)} + C_4(1 - e^{-\alpha(t_{i+1} - s_i)}),\quad i=1, 2,\dots,n \tag{16}$$

Model III

$$(C_1 + C_3\theta)\int_{t_i}^{s_i} e^{\theta(t-t_i)} f(t)dt = \int_{s_{i-1}}^{t_i} [\alpha C_4 + C_2 - \alpha C_2(t_i - t)] e^{-\alpha(t_i - t)} f(t)dt\ , \quad i=1,\dots,n \tag{17}$$

$$\frac{C_1 + C_3\theta}{\theta}[e^{\theta(s_i - t_i)} - 1] = C_2(t_{i+1} - s_i)e^{-\alpha(t_{i+1} - s_i)} + C_4(1 - e^{-\alpha(t_{i+1} - s_i)}),\quad i=1, 2,\dots,n-1 \tag{18}$$

Model IV

$$(C_1 + C_3\theta)\int_{t_i}^{s_i} e^{\theta(t-t_i)} f(t)dt = \int_{s_{i-1}}^{t_i} [\alpha C_4 + C_2 - \alpha C_2(t_i - t)] e^{-\alpha(t_i - t)} f(t)dt\ , \quad i=1,2,\dots,n \tag{19}$$

$$\frac{C_1 + C_3\theta}{\theta}[e^{\theta(s_i - t_i)} - 1] = C_2(t_{i+1} - s_i)e^{-\alpha(t_{i+1} - s_i)} + C_4(1 - e^{-\alpha(t_{i+1} - s_i)}),\quad i=1, 2,\dots,n \tag{20}$$

In the appendix we prove that any solution, t_i^*, s_i^* of the above system of equations satisfies the second order conditions for a minimum. Moreover as we shall see the system of equations {(13), (14)} {(15), (16)} {(17), (18)} {(19), (20)} has a unique solution in the interval [0, H] and so the above minimum is a global minimum.

We shall now present the methodology used to solve the equations (13) and (14), for model I. It is easy to see that, once s_1 is known $t_2(s_1)$ can be obtained from (14). Then $s_2(s_1)$ can be obtained from (13) and following this alternate procedure we can find $t_3(s_1),\ldots, s_n(s_1)$. It is obvious that the optimal replenishment policy that minimizes the total inventory cost, for a given n, requires the selected value of s_1 to be such that $s_n(s_1)=H$ (similar methodology followed for model II).

Similarly, for model III, once t_1 is known, $s_1(t_1)$ can be obtained from (17). Then $t_2(t_1)$ can be obtained from (18) and with same way we can find $s_2(t_1),\ldots, s_n(t_1)$. It is obvious that the optimal replenishment policy that minimizes the total inventory cost, for a given n, requires the selected value of t_1 to be such that $s_n(t_1)=H$ (similar methodology followed for model IV).

The following results guarantees the existence and uniqueness of such value s_1 for models I, II and t_1 for models III, IV. (The proofs referred to model III, for the other models we follow the same steps)

Lemma 1. a) $s_n(0)<H$ and $s_n(H)>H$ (model I, III).

b) $t_{n+1}(0)<H$ and $t_{n+1}(H)>H$ (model II, IV).

Proof. If $t_1=0$, then from (17) we have $s_1=0$. From (18) we have

$$C_2 e^{-\alpha t_2} t_2 + C_4 - C_4 e^{-\alpha t_2} = 0$$

Let us set $h(t_2)= C_2 e^{-\alpha t_2} t_2 + C_4 - C_4 e^{-\alpha t_2}$ then $h(0)=0$ so a root is $t_2^1=0$.

$\frac{dh(t_2)}{dt_2} = e^{-\alpha t_2}(C_2 + \alpha C_4 - \alpha C_2 t_2)$ from this equation we conclude that $h(t_2)$ is increasing function if $t_2<\frac{1}{\alpha}+\frac{C_4}{C_2}$ and decreasing function if $t_2>\frac{1}{\alpha}+\frac{C_4}{C_2}$ this means that if there is a second root t_2^2 it must be greater than $\frac{1}{\alpha}+\frac{C_4}{C_2}$ but $\alpha<\frac{C_2}{C_2 H - C_4}$ so $t_2^2>H$. Therefore, the only acceptable root for equation $C_2 e^{-\alpha t_2} t_2 + C_4 - C_4 e^{-\alpha t_2} = 0$ is $t_2=0$. Following the same way we find $s_2=0$, $t_3=0,\ldots,s_n=0$

b) If $t_1=H$ then from equation (17) follows that $s_1>H$. Equation (18) yields

$\frac{C_1 + C_3\theta}{\theta}[e^{\theta(s_1-H)} - 1] = C_2(t_2 - s_1)e^{-\alpha(t_2-s_1)} + C_4(1-e^{-\alpha(t_2-s_1)})$ we set

$$g(t_2)= \frac{C_1 + C_3\theta}{\theta}[e^{\theta(s_1-H)} - 1] - C_2(t_2 - s_1)e^{-\alpha(t_2-s_1)} - C_4(1-e^{-\alpha(t_2-s_1)})$$

if $t_2=s_1$ then $f(s_1)>0$

$\frac{dg(t_2)}{dt_2}=-(C_2+\alpha C_4-\alpha C(t_2-s_1))e^{-\alpha(t_2-s_1)}$ this means that $g(t_2)$ is a strictly decreasing function of t_2.

Consequently $t_2>s_1$ and so $t_2>H$. Following the same way we find that $s_2>H$, $t_3>H$, …,$s_n>H$.

Lemma 2. a) $s_n(s_1)$ is an increasing function of the variable s_1 (Model I).

b) $t_{n+1}(s_1)$ is an increasing function of the variable s_1 (Model II).

c) $s_n(t_1)$ is an increasing function of the variable t_1 (Model III).

b) $t_{n+1}(t_1)$ is an increasing function of the variable t_1 (Model IV).

Proof. Since f(t) is logconcave, $\frac{f(t)}{f'(t)}$ is strictly increasing in t, for $t_i \le t \le s_i$, and so we have $f'(t)\le\frac{f'(t_i)}{f(t_i)}f(t)$. Multiplying both members of this inequality by $e^{\theta(t-t_i)}$, and adding $\theta\, e^{\theta(t-t_i)} f(t)>0$ to both sides of the resulting equation, we obtain the following inequality

$$e^{\theta(t-t_i)}f'(t)+\theta e^{\theta(t-t_i)}f(t)\le e^{\theta(t-t_i)}\frac{f'(t_i)}{f(t_i)}f(t)+\theta e^{\theta(t-t_i)}f(t),\ t_i\le t\le s_i \tag{21}$$

In (21) if we multiply both sides by $C_1+C_3\theta$ and integrate with respect to t in the interval $[t_i, s_i]$, we have

$$(C_1+C_3\theta)[e^{\theta(s_i-t_i)}f(s_i)-f(t_i)]\le(C_1+C_3\theta)[\frac{f'(t_i)}{f(t_i)}+\theta]\int_{t_i}^{s_i}e^{\theta(t-t_i)}f(t)dt \tag{22}$$

Due to (17) the above inequality becomes

$$(C_1+C_3\theta)(e^{\theta(s_i-t_i)}f(s_i)-f(t_i)-\theta\int_{t_i}^{s_i}e^{\theta(t-t_i)}f(t)dt)\le\frac{f'(t_i)}{f(t_i)}\int_{s_{i-1}}^{t_i}e^{-\alpha(t_i-t)}f(t)(C_2+\alpha C_4-\alpha C_2(t_i-t))dt \tag{23}$$

But $\frac{f'(t_i)}{f(t_i)}<\frac{f'(t)}{f(t)}$ for $s_{i-1}\le t\le t_i$, and $\alpha<\frac{C_2}{C_2H-C_4}$. Taking these into account in the right hand side of (23) and then integrating by parts we obtain

$$\begin{aligned}&(C_1+C_3\theta)[e^{\theta(s_i-t_i)}f(s_i)-f(t_i)-\theta\int_{t_i}^{s_i}e^{\theta(t-t_i)}f(t)dt]\le(C_2+\alpha C_4)f(t_i)\\&-e^{-\alpha(t_i-s_{i-1})}(C_2+\alpha C_4-\alpha C_2(t_i-s_{i-1}))f(s_{i-1})-\int_{s_{i-1}}^{t_i}\alpha e^{-\alpha(t_i-t)}f(t)(2C_2+\alpha C_4-\alpha C_2(t_i-t))dt\end{aligned} \tag{24}$$

Substituting $M_i=s_i-t_i$ and $K_i=t_i-s_{i-1}$ into (17) and differentiating (17) with respect to t_1 (obviously M_i, K_i are functions of t_1) for i=1,…, n, we obtain the following system of equations.

$$(C_1 + C_3\theta)\{e^{\theta M_i} f(s_i)\frac{dM_i}{dt_1} + \frac{dt_i}{dt_1}[e^{\theta M_i} f(s_i) - f(t_i) - \theta\int_{t_i}^{s_i} e^{\theta(t-t_i)} f(t)dt]\} = (C_2 + \alpha C_4)\frac{dt_i}{dt_1} f(t_i)$$
$$-(\frac{dt_i}{dt_1} - \frac{dK_i}{dt_i})(C_2 + \alpha C_4 - \alpha C_2 K_i)e^{-\alpha K_i} f(s_{i-1}) - \alpha\frac{dt_i}{dt_1}\int_{s_{i-1}}^{t_i} e^{-\alpha(t_i - t)} f(t)(2C_2 + \alpha C_4 - \alpha C_2(t_i - t))dt \quad (25)$$

If i=1, then $\frac{dt_1}{dt_1} = 1$ and $\frac{dK_1}{dt_1} = 1$ so

$$(C_1 + C_3\theta)e^{\theta M_1} f(s_1)\frac{dM_1}{dt_1} = \frac{dK_1}{dt_1} e^{-\alpha K_1} f(s_0) + (C_2 + \alpha C_4) f(t_1) - e^{-\alpha K_1}(C_2 + \alpha C_4 - \alpha C_2 K_1) f(s_0)$$
$$-\alpha\int_{s_0}^{t_1} e^{-\alpha(t_1 - t)}(2C_2 + \alpha C_4 - \alpha C_2(t_1 - t)) f(t)dt - (C_1 + C_3\theta)[f(s_1)e^{\theta M_1} - f(t_1) - \theta\int_{t_1}^{s_1} e^{\theta(t-t_1)} f(t)dt] \quad (26)$$

Using (24) and the fact that $\alpha < \frac{C_2}{C_2 H - C_4}$ we see that the right hand side of (26) is a positive number, which implies that $\frac{dM_1}{dt_1} > 0$.

Next, differentiating (18) with respect to t_1 we have

$$(C_1 + C_3\theta)e^{\theta M_1}\frac{dM_1}{dt_1} = (C_2 + \alpha C_4 - \alpha C_2 K_2)\frac{dK_2}{dt_1} e^{-\alpha K_2}$$

The first side of above equation is positive so, $\frac{dK_2}{dt_1} > 0$.

Continuing in this way we can show that $\frac{dK_i}{dt_1} > 0$ and $\frac{dM_i}{dt_1} > 0$ for i=1,…,n. It is obvious that $s_n(t_1) = \sum_{i=1}^{n}(M_i + N_i)$. The above results show that $\frac{ds_n(t_1)}{dt_1} > 0$, which means that $s_n(t_1)$ is strictly increasing function of t_1.

Using induction on i, and following the same procedure we can prove the following.

Corollary 1. a) The functions $t_i(s_1)$, $s_i(s_1)$ are monotonically increasing w.r.t. s_1 (Model I, II).

b) The functions $t_i(t_1)$, $s_i(t_1)$ are monotonically increasing w.r.t. t_1 (Model III, IV).

The above lemmas lead to the following result.

Theorem 1. a) For the model I there exist a unique solution $s_1^* \in [0, H]$ satisfying the $s_n(s_1^*)=H$.

b) For the model II there exist a unique solution $s_1^* \in [0, H]$ satisfying the $t_{n+1}(s_1^*)=H$.

c) For the model III there exist a unique solution $t_1^* \in [0, H]$ satisfying the $s_n(t_1^*)=H$.

d) For the model IV there exist a unique solution $t_1^* \in [0, H]$ satisfying the $t_{n+1}(t_1^*)=H$.

This theorem guarantees the uniqueness of the optimal replenishment schedule for any policy with n replenishments.

Now we present a theorem, which ensures the existence of a unique optimal replenishment number, n, and consequently the existence of a unique optimal policy for the problem under consideration.

Theorem 2. The functions $TC_j(n, s_i, t_i)$ $j=1, 2, 3, 4$ are convex w.r.t. n

Proof. The technique used in the proof of this theorem involves dynamic programming arguments and is similar to that used by Teng et al.[1], Friedman[14]. Let as set

$$TC(n, s_i, t_i)=nC+T(n, s_i, t_i) \tag{27}$$

where

$$T(n, s_i, t_i)=\frac{C_1+C_3\theta}{\theta}\sum_{i=0}^{n-1}(e^{\theta(t-t_{i+1})}-1)f(t)dt+\sum_{i=0}^{n-1}\int_{s_i}^{t_{i+1}}(C_4-C_4e^{-\alpha(t_{i+1}-t)}+C_2e^{-\alpha(t_{i+1}-t)}(t_{i+1}-t))f(t)dt$$

It is enough to prove that

$$T(n+1, 0, H)-T(n, 0, H)>T(n, 0, H)-T(n-1, 0, H) \tag{28}$$

By Bellman[15] 's principle of optimality, we obtain the minimum value of $T(n, s_i, t_i)$

$$T^*(n, 0, H)=\min_{s\in[0,H]}\{T^*(n-1, 0, s)+T(1, s, H)\} \tag{29}$$

Recursive application of (29) yields the optimal *i*th shortage point, $s_i^*(n, 0, H)=s_i^*(n\text{-}j, 0, s_{n-j}^*(n, 0, H))$ $i=1,\dots,n\text{-}j\text{-}1$, when n orders placed in the interval [0, H].

Let s=H and hence $T^*(n, 0, H)<T^*(n-1, 0, H)$. Thus, $T^*(n, 0, H)$ is strictly decreasing on n. Let choose H_1 and H_2 such that

$$s_n^*(n+1, 0, H_1)=s_{n+1}^*(n+2, 0, H_2)=H \tag{30}$$

Since $s_n^*(n+1, 0, H_1)=H$ employing the principle of optimality, we have

$$T^*(n+1, 0, H_1)=\min_{s\in[0,H_1]}\{T^*(n, 0, s)+T(1, s, H_1)\}$$

$$=T^*(n, 0, H)+T(1, H, H_1) \tag{31}$$

But this means that if t=H then

$$\frac{\partial[T^*(n,0,t)+T(1,t,H_1)]}{\partial t}=\frac{\partial T^*(n,0,t)}{\partial t}+\frac{\partial T(1,t,H_1)}{\partial t}=0 \tag{32}$$

So

$$\frac{\partial T^*(n,0,t)}{\partial t}\Big|_{t=H}=-\frac{\partial T(1,t,H_1)}{\partial t}\Big|_{t=H}$$

$$=\frac{C_1+C_3\theta}{\theta}(e^{\theta(H-t_n^*(n,0,H))}-1))f(H) \tag{33}$$

where t_n^* (n, 0, H) is the last optimal replenishment time when n orders placed during the interval [0, H].

Similarly, from s_{n+1}^* (n+2, 0, H_2)=H, we have

$$\frac{\partial T^*(n+1,0,t)}{\partial t}\Big|_{t=H}=-\frac{\partial T(1,t,H_1)}{\partial t}\Big|_{t=H}$$

$$=\frac{C_1+C_3\theta}{\theta}(e^{\theta(H-t_{n+1}^*(n+1,0,H))}-1))f(H) \tag{34}$$

Subtracting eq. (34) from eq. (33) we take

$$\frac{\partial[T^*(n,0,H)-T^*(n+1,0,H)]}{\partial H}>0 \tag{35}$$

which implies that T^*(n, 0, H)-T^*(n+1, 0, H) is a strictly increasing function of H. Using (29) and (31) we obtain

$$T^*(n,0,H_1)-T^*(n+1,0,H_1)=\min_{s\in[0,H_1]}\{T^*(n-1,0,s)+T(1,s,H_1)\}$$

$$-T^*(n,0,H)-T(1,H,H_1) \tag{36}$$

Taking s=H in eq. (36) we have

$$T^*(n,0,H_1)-T^*(n+1,0,H_1)<T^*(n-1,0,H)-T^*(n,0,H) \tag{37}$$

Since H<H_1 and T^*(n, 0, H)-T^*(n+1, 0, H) is a strictly increasing function in H, we obtain relation (28). This implies that TC(n, s_i, t_i) is also convex in n.

5. Cost comparison

In this section we shall make comparisons between the optimal total relevant costs of the above four models.

Let us denote these cost by $TC_j(n^j, s_i^j, t_i^j)$, j=1, 2, 3, 4, for models I, II, III, IV respectively.

Theorem 3. $TC_1(n^1, s_i^1, t_i^1) > TC_2(n^2, s_i^2, t_i^2) > TC_4(n^4, s_i^4, t_i^4)$ and

$$TC_1(n^1, s_i^1, t_i^1) > TC_3(n^3, s_i^3, t_i^3) > TC_4(n^4, s_i^4, t_i^4)$$

Proof. For any n from eq. (8), (9) and (15) we have

$$TC_1(n, s_i^1, t_i^1) = TC_2(n, s_i^1, t_i^1) + \frac{C_1 + C_3\theta}{\theta}\int_{s_n}^{H}(e^{\theta(t-t_n)} - 1)f(t)dt - \int_{s_n}^{H}(C_4 - C_4e^{-\alpha(H-t)} + C_2e^{-\alpha(H-t)}(H-t)f(t)dt$$

$$> TC_2(n, s_i^1, t_i^1) + \frac{C_1 + C_3\theta}{\theta}\int_{s_n}^{H}(e^{\theta(s_n-t_n)} - 1)f(t)dt - \int_{s_n}^{H}(C_4 - C_4e^{-\alpha(H-t)} + C_2e^{-\alpha(H-t)}(H-t)f(t)dt$$

$$= TC_2(n, s_i^1, t_i^1) + \int_{s_n}^{H}(C_4 - \alpha C_4e^{-\alpha(H-s_n)} + C_2e^{-\alpha(H-s_n)}(H-s_n)f(t)dt$$

$$- \int_{s_n}^{H}(C_4 - \alpha C_4e^{-\alpha(H-t)} + C_2e^{-\alpha(H-t)}(H-t)f(t)dt$$

$$> TC_2(n, s_i^1, t_i^1) + \int_{s_n}^{H}(C_4 - \alpha C_4e^{-\alpha(H-s_n)} + C_2e^{-\alpha(H-s_n)}(H-s_n)f(t)dt$$

$$- \int_{s_n}^{H}(C_4 - \alpha C_4e^{-\alpha(H-t)} + C_2e^{-\alpha(H-t)}(H-t)f(t)dt$$

$$> TC_2(n, s_i^1, t_i^1) + \int_{s_n}^{H}(C_4 - \alpha C_4e^{-\alpha(H-s_n)} + C_2e^{-\alpha(H-s_n)}(H-s_n)f(t)dt$$

$$- \int_{s_n}^{H}(C_4 - \alpha C_4e^{-\alpha(H-s_n)} + C_2e^{-\alpha(H-s_n)}(H-s_n)f(t)dt$$

$$= TC_2(n, s_i^1, t_i^1) > TC_2(n, s_i^2, t_i^2)$$

If n^1 is the optimal number of replenishment for model I then

$$TC_1(n^1, s_i^1, t_i^1) > TC_2(n^1, s_i^2, t_i^2) \geq TC_2(n^2, s_i^2, t_i^2)$$

Using the same reasoning we obtain

$$TC_3(n^3, s_i^3, t_i^3) > TC_4(n^4, s_i^4, t_i^4)$$

Now we shall prove that $TC_2(n^2, s_i^2, t_i^2) > TC_4(n^4, s_i^4, t_i^4)$

For any n we have

$$TC_2(n, s_i^2, t_i^2) = \frac{C_1 + C_3\theta}{\theta}\int_0^{s_1}(e^{\theta t} - 1)f(t)dt - \frac{C_1 + C_3\theta}{\theta}\int_{t_1}^{s_1}(e^{\theta(t-t_1)} - 1)f(t)dt$$

$$-\int_0^{t_1}(C_4 - C_4 e^{-\alpha(t_1-t)} + C_2(t_1 - t)e^{-\alpha(t_1-t)}f(t)dt$$

Now we will show that the quantity

$$g(t_1) = \frac{C_1 + C_3\theta}{\theta}\int_0^{s_1}(e^{\theta t} - 1)f(t)dt - \frac{C_1 + C_3\theta}{\theta}\int_{t_1}^{s_1}(e^{\theta(t-t_1)} - 1)f(t)dt$$

$$-\int_0^{t_1}(C_4 - C_4 e^{-\alpha(t_1-t)} + C_2(t_1 - t)e^{-\alpha(t_1-t)}f(t)dt$$

is nonnegative.

Since $g(0)=0$, it 's enough to show that $g(t_1)$ is increasing function of t_1.

This is true because

$$\frac{dg(t_1)}{dt_1} = \frac{C_1 + C_3\theta}{\theta}\frac{ds_1}{dt_1}e^{\theta(s_1-t_1)}f(s_1)(e^{\theta t_1} - 1) > 0$$

and so we have $TC_2(n, s_i^2, t_i^2) > TC_4(n, s_i^4, t_i^4)$.

If n^2 is the optimal number of replenishment for model II then

$TC_2(n^2, s_i^2, t_i^2) > TC_4(n^2, s_i^4, t_i^4) \geq TC_4(n^4, s_i^4, t_i^4)$

In a similar way we can prove that $TC_1(n^1, s_i^1, t_i^1) > TC_3(n^3, s_i^3, t_i^3)$. This completes the proof of the theorem.

Example

In order to illustrate the preceding theory we consider the following data

$f(t)=20+2t$, $C=100$, $C_1=0.5$, $C_3=5$, $\theta=0.01$, $\alpha=0.08$, $H=12$.

In tables 1, 2, 3, 4 we give the overall optimal cost for the four models for different values of α, C_2 and C_4.

We notice that if $C_4=0$ the total cost function is a decreasing function of the parameter α, as follows easily from (9), (10), 11) and (12). This implies that, if we assume no cost of lost sales, the presenting models with this type of partial backlogging has always smaller total cost than the corresponding models with complete backlogging of unsatisfied demand. If we assume cost due to lost sales we have not a general result.

6. Concluding remarks

In this paper, we have presented four inventory models and we have obtained their respective optimal replenishment policies. The basic contribution of the paper is the introduction of time variable partial backlogging of unsatisfied demand, which gives flexibility to the models. Moreover we have made comparisons between their optimal total relevant costs. This comparison has shown that model IV is the least expensive among all four. It is also the most flexible, in the sense that does not require an order at the beginning of the planning horizon. The models studied here and the results obtained contain as special cases, for $\alpha=0$, the models and the results obtained by Teng et al.[1] when the rate of demand, $f(t)$, is a logconcave function of t.

Appendix

Checking the conditions for a minimum of $TC(n, s_i, t_i)$.

For convenience let us $TC(n, s_i, t_i)=TC$. To ensure that the solution of equation (17) and (18) gives a minimum it is enough to prove that the associated Hessian matrix has positive principal minors. The elements of this Hessian matrix are.

$$H_{2k+1,2j+1}=\frac{\partial^2 TC}{\partial t_{k+1}\partial t_{j+1}} \quad j, k=0, 1,\dots, n-1,$$

$$H_{2k,2j}=\frac{\partial^2 TC}{\partial s_k \partial s_j} \quad k, j=1,\dots,n-1,$$

$$H_{2k+1,2j}=\frac{\partial^2 TC}{\partial t_{k+1}\partial s_j} \quad k=0, 2, \dots, n-1, j=1,\dots,n,$$

$$H_{2k,2j+1}=\frac{\partial^2 TC}{\partial s_k \partial t_{j+1}} \quad k=1,\dots,n, j=1,2, \dots, n-1.$$

The non - zero entries of the Hessian matrix are

$$H_{2j+1,2j+1}=\frac{\partial^2 TC}{\partial t_{j+1}^2}=$$

$$(C_1+C_3\theta)f(t_{j+1})+\theta(C_1+C_3\theta)\int_{t_{j+1}}^{s_{j+1}} e^{\theta(t-t_{j+1})}f(t)dt+(C_2+\alpha C_4)f(t_{j+1})$$

$$-\alpha\int_{s_j}^{t_{j+1}} e^{-\alpha(t_{j+1}-t)}(2C_2+\alpha C_4-\alpha C_2(t_{j+1}-t)f(t)dt \quad j=0,\dots,n-1,$$

$$H_{2j,2j}=\frac{\partial^2 TC}{\partial s_j^2}= [(C_1+C_3\theta)e^{\theta(s_j-t_j)}+(C_2+\alpha C_4-\alpha C_2(t_{j+1}-s_j))e^{-\alpha(t_{j+1}-s_j)}]f(s_j) \quad j=1,2,\dots,n,$$

$$H_{2k+1,2j}=H_{2j,2k+1}=\frac{\partial^2 TC}{\partial t_{k+1}\partial s_j}= -(C_1+C_3\theta)e^{\theta(s_j-t_{k+1})}f(s_j) \quad k=0, 2,\dots, n-1, j=k+1,$$

$$H_{2j+1,2k}=H_{2k,2j+1}=\frac{\partial^2 TC}{\partial s_k \partial t_{j+1}}=-e^{-\alpha(t_{j+1}-s_k)}(C_2+\alpha C_4-\alpha C_2(t_{j+1}-s_k))f(s_k) \quad k=1, \dots, n-1, \quad j=k.$$

We observe that $H_{2j+1,2j+1}>0$, $H_{2j,2j}>0$, $H_{2k+1,2j}<0$ and $H_{2k,2j+1}<0$

Let M_k be the principal minor of order k, then

$$M_1=H_{1,1}>(C_1+C_3\theta)e^{\theta(s_j-t_j)}f(s_j)+(C_2+\alpha C_4-\alpha C_2(t_j-s_{j-1}))e^{-\alpha(t_j-s_{j-1})}f(s_{j-1})>0.$$

This follows using the inequality (23).

$$M_2=\frac{\partial^2 TC}{\partial s_1^2}\frac{\partial^2 TC}{\partial t_1^2}-\frac{\partial^2 TC}{\partial s_1\partial t_1}\frac{\partial^2 TC}{\partial t_1\partial s_1}>M_1\, e^{-\alpha(t_1-s_0)}\,(C_2+\alpha\, C_4-\alpha C_2(t_1-s_0))[f(s_1)]+$$

$$(C_1+C_3\theta)\,e^{\theta(s_1-t_1)}\,e^{-\alpha(t_1-s_0)}\,(C_2+\alpha\, C_4-\alpha C_2(t_1-s_0))f(s_1)\,f(s_0)>0$$

or, equivalently

$$M_2+\frac{\partial^2 TC}{\partial t_2\partial s_1}M_1>(C_1+C_3\theta)\,e^{\theta(s_1-t_1)}\,e^{-\alpha(t_1-s_0)}\,(C_2+\alpha\, C_4-\alpha C_2(t_1-s_0))f(s_1)f(s_0)>0$$

So we have shown that $M_1>0$, $M_2>0$ and $M_2+\frac{\partial^2 TC}{\partial t_2\partial s_1}M_1>0$

It is not difficult to verify that the principal minors of higher order satisfy the following recurrence relationships

$$M_{2j+1}=\frac{\partial^2 TC}{\partial t_{j+1}^2}M_{2j}-\left(\frac{\partial^2 TC}{\partial t_{j+1}\partial s_j}\right)^2 M_{2j-1},\ j=1,2,\ldots,n-1, \tag{A1}$$

$$M_{2j}=\frac{\partial^2 TC}{\partial s_j^2}M_{2j-1}-\left(\frac{\partial^2 TC}{\partial t_j\partial s_j}\right)^2 M_{2j-2},\ j=2,\ldots,n. \tag{A2}$$

Also we observe that

$$\frac{\partial^2 TC}{\partial t_{j+1}^2}>-\frac{\partial^2 TC}{\partial t_{j+1}\partial s_j}-\frac{\partial^2 TC}{\partial t_{j+1}\partial s_{j+1}}\quad\text{and}\quad\frac{\partial^2 TC}{\partial s_{j+1}^2}=-\frac{\partial^2 TC}{\partial t_{j+1}\partial s_j}-\frac{\partial^2 TC}{\partial t_{j+1}\partial s_{j+1}}.$$

From (A1) and (A2) we obtain

$$M_{2j+1}+\frac{\partial^2 TC}{\partial t_{j+1}\partial s_{j+1}}M_{2j}>-\frac{\partial^2 TC}{\partial t_{j+1}\partial s_j}\left(M_{2j}+\frac{\partial^2 TC}{\partial t_{j+1}\partial s_j}M_{2j-1}\right), \tag{A3}$$

$$M_{2j}+\frac{\partial^2 TC}{\partial t_{j+1}\partial s_j}M_{2j-1}=-\frac{\partial^2 TC}{\partial t_j\partial s_j}\left(M_{2j-1}+\frac{\partial^2 TC}{\partial t_j\partial s_j}M_{2j-2}\right). \tag{A4}$$

Now, by applying inequality (A3) for $j=1$, we get $M_3>0$. This result, accompanied by equation (A4) for $j=2$ results to $M_4>0$. Using induction we can prove that the principal minors of any order are positive.

	$\alpha=0$	$\alpha=0.08$	$\alpha=1.4$
$C_2=1.5$, $C_4=0.5$	762.556	762.172	
$C_2=1.5$, $C_4=1.5$	762.556	764.345	
$C_2=1.5$, $C_4=0$	762.556	761.066	
$C_2=1.5$, $C_4=17$	762.556	783.538	811.766
$C_2=4.5$, $C_4=0.5$	795.606	793.997	
$C_2=6.5$, $C_4=0$	800.843	799.942	

Table 1. Optimal cost for model I for different values α, C_2 and C_4.

	$\alpha=0$	$\alpha=0.08$	$\alpha=1.4$
$C_2=1.5$, $C_4=0.5$	755.212	754.938	
$C_2=1.5$, $C_4=1.5$	755.212	757.114	
$C_2=1.5$, $C_4=0$	755.212	754.712	
$C_2=1.5$, $C_4=17$	755.212	780.938	810.204
$C_2=4.5$, $C_4=0.5$	791.43	791.418	
$C_2=6.5$, $C_4=0$	798.65	798.368	

Table 2. Optimal cost for model II for different values α, C_2 and C_4.

	$\alpha=0$	$\alpha=0.08$	$\alpha=1.4$
$C_2=1.5$, $C_4=0.5$	754.872	756.05	
$C_2=1.5$, $C_4=1.5$	754.872	759.579	
$C_2=1.5$, $C_4=0$	754.872	753.938	
$C_2=1.5$, $C_4=17$	754.872	778.483	811.494
$C_2=4.5$, $C_4=0.5$	792.336	791.336	
$C_2=6.5$, $C_4=0$	800.634	798.695	

Table 3. Optimal cost for model III for different values α, C_2 and C_4.

	$\alpha=0$	$\alpha=0.08$	$\alpha=1.4$
$C_2=1.5$, $C_4=0.5$	749.727	750.54	
$C_2=1.5$, $C_4=1.5$	749.727	750.576	
$C_2=1.5$, $C_4=0$	749.727	748.039	
$C_2=1.5$, $C_4=17$	749.727	775.453	809.644
$C_2=4.5$, $C_4=0.5$	789.337	789.875	
$C_2=6.5$, $C_4=0$	797.654	794.411	

Table 4. Optimal cost for model IV for different values α, C_2 and C_4.

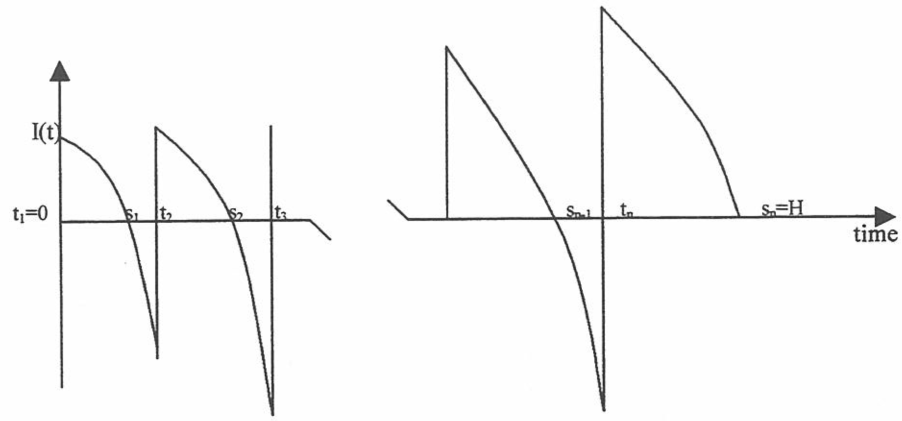

Figure 1. Graphical representation of inventory Model I

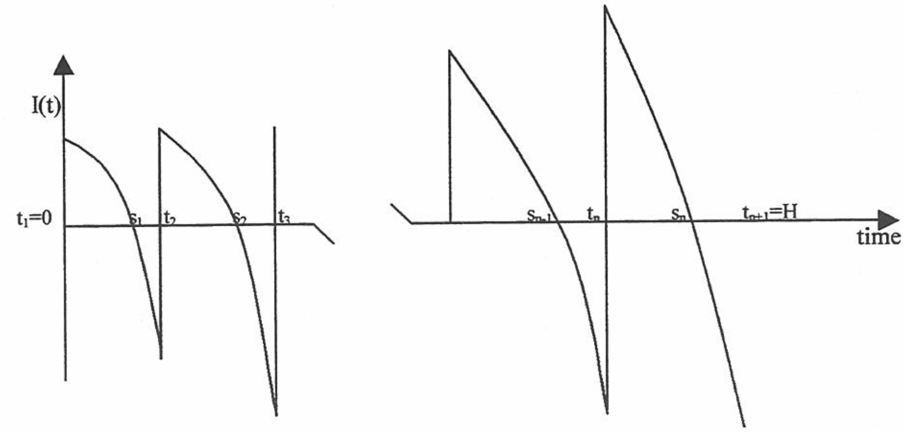

Figure 2. Graphical representation of inventory Model II

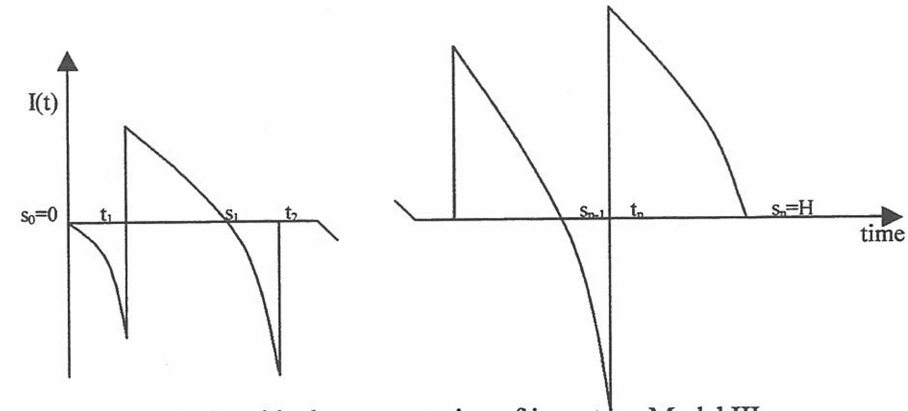

Figure 3. Graphical representation of inventory Model III

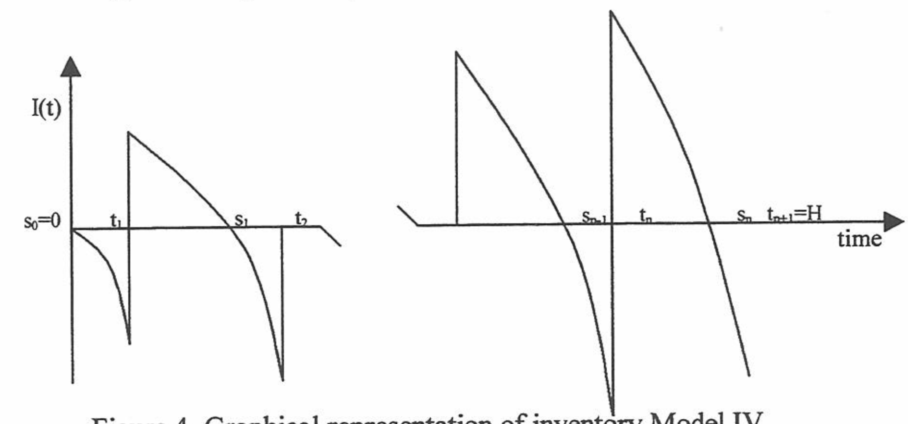

Figure 4. Graphical representation of inventory Model IV

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SELECTION OF THE BEST POPULATION: AN INFORMATION THEORETIC APPROACH

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Abstract

This paper is devoted to the statistical problem of ranking and selection populations by using the subset selection formulation. The interest is focused (i) on the selection of the best population among k independent populations and (ii) on the selection of the best population, which is closest to an additional standard or control population. With respect to the first problem the populations are ranked in terms of entropies of their distributions and the population whose distribution has maximum entropy is selected. For the second problem the populations are ranked in terms of divergences of their distributions and the distribution of the standard or control population. The population with the minimum divergence is selected. In each case the populations are assumed to have general parametric densities satisfying the classical regularity conditions of asymptotic statistic. Large sample properties of the estimators of entropies and divergences of the populations will be studied and used in order to determine the probabilities of correct selection of the proposed asymptotic selection rules. Illustrative examples, including a numerical example using real medical data, will be given for multivariate homoscedastic normal populations and populations described by the regular exponential family of distributions.

Keywords and phrases: Divergence; Entropy; Exponential family; Mahalanobis distance; Multivariate normal distribution; Ordering populations; Selection of populations; Subset selection approach.

1 Introduction

This paper deals with statistical problems of ranking and selection. It is of great theoretical and practical importance to build statistical methodology that helps to analyze data from k independent experiments that have been subject to different (medical, agricultural, manufactoring etc.) treatments. The classical test procedure known as the analysis of variance was introduced by Fisher as a method dealing with the question of whether the k treatment effects are different or not. In many experiments it seems obvious that the different treatments must have produced some difference, however small in effect. Then the main problem is to find which of the k treatments effects is the best or the largest in some sense. Experiments will be called populations, in the sequel, while treatment effects will be the associated parameters. For the solution of this problem two classical approaches to ranking and selection have been developed during the last fifty years by several authors: namely, the indifference zone approach of Bechhofer (1954) and the subset selection formulation of Gupta (1965). These general approaches are well known and discussions on them can be found in the books by Gibbons et al. (1977) and Gupta and Panchapakesan (1979). We also refer, among other, the following papers devoted to the subject selection approach which will be of our interest in the sequel: Gupta (1963, 1966), Gupta and Studden (1970), Gupta and Panchapakesan (1972), Panchapakesan and Santner (1977), Chattopadhyay (1981). Chen and Sobel (1987), and Jeyaratnam and Panchapakesan (1997) considered an integrated formulation of the problem combining features of the two classical approaches, while some other papers with slightly modified goal are referred by them. Another methodological approach is given by Liese and Miescke (1999) who study the problem of selecting the best of k populations with methods from asymptotic decision theory. Finally we refer to the paper by Thabane and Safiul Haq (1999) where the same problem is studied by using the Bayesian approach.

Typical situations encountered in these and other treatments are as follows: We have $k \geq 2$ independent parametric populations and on the basis of random samples from these populations we wish to select the best population or the t, $2 \leq t \leq k$, best populations or to select the best population closest to an additional standard or control population. The term best population means the largest (or, in some cases, smallest) population that results from a ranking procedure of the said populations. For most univariate problems the selection and ranking has been defined in terms of location or scale parameters. Relatively limited is the literature on selection procedures for multivariate populations. In this case the interest was mainly focused on multivariate normal populations and one is usually interested in the ranking and selection problems in terms of suitably defined functions of the several parameters. These functions are usually some scalar quantities and the Mahalanobis distance from the origin is the most commonly used function for selecting and ranking the best from k multivariate normal populations.

In this paper we are concerned with subset selection procedures for multivariate populations. We will discuss selection of the best of k populations and selection of the best population, which is closest to an additional standard or control population. The populations considered will be members of a general multiparameter family of distributions. The interest will also be focused on the multiparameter exponential family of distributions as well as the multivariate normal distribution. Ranking and selection procedures from several populations will be based on generalized distance measures defined in the field of statistical information. Entropy and divergence measures will serve as functions of the parameters of the populations considered. In this context, in the next section, ranking and selection rules will be stated in terms of entropy or divergence functions of the populations. Section 3 will be devoted to the large sample behaviour of the estimators of entropy and divergence functions, which will be obtained if the unknown parameters of the respective populations are replaced by their maximum likelihood estimators. The asymptotic results of Section 3 will help in order to determine, in the next section, the probabilities of correct selection of the selection rules. In the final Section 5 the ranking and selection procedures will be illustrated for multivariate homoscedastic normal populations, for populations described by the regular exponential family of distributions as well as the gamma distribution.

2 Statement of the problem and notations

Suppose that $\{f_\theta : \theta \in \Theta\}$ is a family of probability densities with respect to a measure μ, defined on the space $\mathfrak{X}$, and the parameter space $\Theta \subset R^M$ is an open set. Suppose also that the classical regularity assumptions $(R1) - (R3)$ appearing in pages 144-145 of Serfling (1980) are satisfied:

$(R1)$ For each $\theta \in \Theta$, f_θ is positive and the derivatives

$$\frac{\partial \log f_\theta}{\partial \theta_s}, \quad \frac{\partial^2 \log f_\theta}{\partial \theta_s \partial \theta_r}, \quad \frac{\partial^3 \log f_\theta}{\partial \theta_s \partial \theta_r \partial \theta_t},$$

exist everywhere on $\mathfrak{X}$, for all $1 \leq s, r, t \leq M$.

$(R2)$ For each $\theta_0 \in \Theta$, there exist measurable functions $\alpha, \beta, \gamma : \mathfrak{X} \to [0, \infty)$, possibly depending on θ_0, such that, for all θ in a neighborhood $N(\theta_0)$ and all above considered s, r, t, the relations

$$\left|\frac{\partial f_\theta}{\partial \theta_s}\right| \leq \alpha, \quad \left|\frac{\partial^2 f_\theta}{\partial \theta_s \partial \theta_r}\right| \leq \beta, \quad \left|\frac{\partial^3 \log f_\theta}{\partial \theta_s \partial \theta_r \partial \theta_t}\right| \leq \gamma,$$

hold on $\mathfrak{X}$, and

$$\int_{\mathfrak{X}} \alpha d\mu < \infty, \quad \int_{\mathfrak{X}} \beta d\mu < \infty, \quad \int_{\mathfrak{X}} \gamma f_\theta d\mu < \infty.$$

$(R3)$ For each $\theta \in \Theta$, the Fisher information matrix

$$I(\theta) = \left[\int_{\mathfrak{X}} \frac{\partial \log f_\theta}{\partial \theta_s} \frac{\partial \log f_\theta}{\partial \theta_r} f_\theta d\mu\right]_{M \times M},$$

exists and is positive definite, with all elements continuous in the variable θ.

Further assume that $\theta_i = (\theta_{i1}, ..., \theta_{iM})^t$ denotes the parameter of the ith population Π_i, with density f_{θ_i}, $i = 1, ..., k$, and that the following identifiability condition is satisfied:

$(R4)$ If $f_{\theta_i} = f_{\theta_j}$, *a.e.* μ, then $\theta_i = \theta_j$, $i, j = 1, ..., k$.

Here, and in the sequel, "t" denotes the vector or matrix transpose.

There is an extensive literature and many tests for testing the homogeneity hypothesis $H_0: \ \theta_1 = \theta_2 = ... = \theta_k$. However, in practice, a test of homogeneity is of limited interest and usually alternative methods of statistical analysis are needed. For example, the statistician may be interested to determine (i) which of the k populations differ from which others or (ii) to see which populations can be considered the best in some well defined sense of the term "best". Multiple comparisons methods are useful procedures for solving the first problem, while ranking and selection procedures are more appropriate for the solution of the second problem. We will focus our interest on ranking and selection procedures of the best of k populations and selection of the best population, which is closest to an additional standard or control population Π_0 with density f_{θ_0}, for $\theta_0 \in \Theta$. The ranking and selection procedures will be based on suitably defined functions of the parameters $\theta_i = (\theta_{i1}, ..., \theta_{iM})^t$, $i = 0, 1, ..., k$.

Denote by $H(\theta)$ the well-known Shannon entropy of the parametric density f_θ, $\theta \in \Theta$, defined by

$$H(\theta) = -\int_{\mathfrak{X}} f_\theta(x) \log f_\theta(x) d\mu. \tag{1}$$

The entropy of the family of densities f_θ is a measure of variation of the uncertainty for the distributions belonging to the family $\{f_\theta : \theta \in \Theta\}$. A more general measure of entropy with the same meaning and similar properties has been defined by Burbea and Rao (1982). This measure is known as $\varphi-$entropy and it is defined by the relation

$$H^\varphi(\theta) = \int_{\mathfrak{X}} \varphi\left(f_\theta(x)\right) d\mu, \tag{2}$$

where φ is a real concave function defined on $[0, \infty)$. In what follows we will suppose that $H^\varphi(\theta) \geq 0$. Shannon's entropy $H(\theta)$, defined by (1), can be immediately obtained from $H^\varphi(\theta)$ for $\varphi(u) = -u \log u$, $u > 0$. Due to the meaning of $H(\theta)$ or $H^\varphi(\theta)$, the ranking of populations Π_i, in terms of their Shannon or $\varphi-$entropies, can be intuitively interpreted as a ranking of the variation of the diversity of the respective populations. Let $H_i^\varphi = H^\varphi(\theta_i)$ be the entropy of the ith population Π_i, with $i = 1, ..., k$, and let H^φ's be ranked as

$$H_{[1]}^\varphi \leq H_{[2]}^\varphi \leq ... \leq H_{[k]}^\varphi.$$

Suppose that there is not any apriori information regarding the correct pairing of the ordered $H_{[i]}^\varphi$ with the k given populations. In this context, any population associated with $H_{[k]}^\varphi$ will be called the best population. Dudewicz and van der Meulen (1983) have developed methods for the selection of the best entropy population for continuous distributions on (0,1), while Gupta and Panchapakesan (1987) have used Shannon entropy for the selection from several multinomial populations.

Consider now the selection of the best population, among $\Pi_1, ..., \Pi_k$, which is closest to the population Π_0 with density f_{θ_0}, for $\theta_0 \in \Theta$. The parameter θ_0 may be known or unknown. If θ_0 is known, Π_0 is referred to as a standard population, while if θ_0 is unknown, Π_0 is called a control population. The ranking and selection formulation in this case is much more intuitively clear. Let us denote by D_i a distance measure between the population Π_i with density f_{θ_i} and the control or standard population Π_0 with density

f_{θ_0}, for θ_0, $\theta_i \in \Theta$ and $i = 1, ..., k$. A general class of distance measures, in a broad sense, between two distributions is the well-known ϕ-divergence introduced independently by Csiszar (1963) and Ali and Silvey (1966). The ϕ-divergence between population Π_i and the population Π_0 is defined by

$$D_i^\phi = D^\phi(f_{\theta_i}, f_{\theta_0}) = \int_{\mathfrak{X}} f_{\theta_0}(x)\phi\left(\frac{f_{\theta_i}(x)}{f_{\theta_0}(x)}\right) d\mu, \tag{3}$$

and it is intuitively intended as a measure of the distance, from a statistical point of view, between the respective populations. The subscript *st* or *co* will be used in the notation D_i^ϕ, i.e. $D_{i,st}^\phi$ or $D_{i,co}^\phi$, to declare that the population Π_0 is standard or control, respectively. Let $D^{\phi\prime}$s be ranked as

$$D_{[1],\zeta}^\phi \leq D_{[2],\zeta}^\phi \leq ... \leq D_{[k],\zeta}^\phi,$$

with ζ being *st* if Π_0 is standard and ζ being *co* if Π_0 is a control population. Assume that there is not any apriori information regarding the correct pairing of the ordered $D_{[i],\zeta}^\phi$ with the k given populations. In this context, any population associated with $D_{[1],co}^\phi$or $D_{[1],st}^\phi$ will be called the best population, closest to the control or standard population Π_0.

The $\varphi-$entropies H_i^φ and the ϕ-divergences D_i^ϕ depend on the populations parameters θ_0 and θ_i, $i = 1, ..., k$. Since the parameters θ_i, $i = 1, ..., k$, are unknown it seems reasonable to compute an estimator $\widehat{\theta}_i$ of each θ_i from a corresponding random sample. In a similar context we consider an estimator $\widehat{\theta}_0$ of the unknown parameter θ_0 for the case of a control population Π_0. Based on them the unknown $\varphi-$entropies and ϕ-divergences H_i^φ and $D_{i,\zeta}^\phi$, $i = 1, ..., k$, respectively, can be estimated by the quantities $\widehat{H}_i^\varphi$ and $\widehat{D}_{i,\zeta}^\phi$, $i = 1, ..., k$, if the unknown parameters θ_i and maybe θ_0 are replaced by their estimators $\widehat{\theta}_i$, $i = 1, ..., k$, and $\widehat{\theta}_0$, in the expressions of the respective measures defined by relations (2) and (3). In this direction, consider independent random samples $X_{i1}, X_{i2}, ..., X_{in_i}$ from the population described by f_{θ_i}, , $i = 1, ..., k$, and also an independent random sample $X_{01}, X_{02}, ..., X_{0n_0}$ from the control population f_{θ_0}. Denote by $\widehat{\theta}_0$ and $\widehat{\theta}_i$ the maximum likelihood estimators of the parameters θ_0 and θ_i, $i = 1, ..., k$, respectively. Based on them, the estimators $\widehat{H}_i^\varphi$, $\widehat{D}_{i,st}^\phi$ and $\widehat{D}_{i,co}^\phi$ of H_i^φ, $D_{i,st}^\phi$ and $D_{i,co}^\phi$, $i = 1, ..., k$, are defined by

$$\widehat{H}_i^\varphi = H^\varphi(\widehat{\theta}_i) = \int_{\mathfrak{X}} \varphi\left(f_{\widehat{\theta}_i}(x)\right) d\mu, \tag{4}$$

$$\widehat{D}_{i,st}^\phi = D^\phi(f_{\widehat{\theta}_i}, f_{\theta_0}) = \int_{\mathfrak{X}} f_{\theta_0}(x)\phi\left(\frac{f_{\widehat{\theta}_i}(x)}{f_{\theta_0}(x)}\right) d\mu, \tag{5}$$

and

$$\widehat{D}_{i,co}^\phi = D^\phi(f_{\widehat{\theta}_i}, f_{\widehat{\theta}_0}) = \int_{\mathfrak{X}} f_{\widehat{\theta}_0}(x)\phi\left(\frac{f_{\widehat{\theta}_i}(x)}{f_{\widehat{\theta}_0}(x)}\right) d\mu. \tag{6}$$

In relation with the function ϕ considered in the general $\phi-$divergence statistics, we consider the following assumptions:

(ϕ1) The function $\phi : [0, \infty) \to (-\infty, \infty]$ is convex and continuous. Its restriction on $(0, \infty)$ is finite, twice continuously differentable, with $\phi(1) = \phi'(1) = 0$ and $\phi''(1) = 1$.

(ϕ2) For every $\theta^* \in \Theta$ there exists an open neighborhood $N(\theta^*)$, such that, for all $\theta_i \in N(\theta^*)$, $i = 1, ..., k$, it holds

$$\frac{\partial}{\partial \theta_{ir}} \int_{\mathfrak{X}} f_{\theta^*}(x) \phi\left(\frac{f_{\theta_i}(x)}{f_{\theta^*}(x)}\right) d\mu = \int_{\mathfrak{X}} \frac{\partial}{\partial \theta_{ir}} \left[f_{\theta^*}(x) \phi\left(\frac{f_{\theta_i}(x)}{f_{\theta^*}(x)}\right) d\mu \right], \quad i = 1, ..., k, \; r = 1, ..., M.$$

Condition (ϕ1) deals with properties of $\phi-$divergences (cf. Liese and Vajda (1987)). Condition (ϕ2) is needed to apply delta method for obtaining asymptotic distributions of $\phi-$divergence statistics. Sufficient conditions for the validity of (ϕ2) are presented in Morales et al. (1997).

In this context, it seems reasonable to select the population which gives rise to $\widehat{H}^{\varphi}_{[k]} = \max\{\widehat{H}^{\varphi}_1, ..., \widehat{H}^{\varphi}_k\}$ as the best one or to select the population which gives rise to $\widehat{D}^{\phi}_{[1],\zeta} = \min\{\widehat{D}^{\phi}_{1,\zeta}, ..., \widehat{D}^{\phi}_{k,\zeta}\}$, with ζ being *st* or *co* for the problem of selection of the best population, which is closest to a standard or control population, respectively. However, since the sample from the population with $H^{\varphi}_{[k]}$ and $D^{\phi}_{[1],\zeta}$ do not always produce respectively $\widehat{H}^{\varphi}_{[k]}$ and $\widehat{D}^{\phi}_{[1],\zeta}$, due to sampling variation, we need a probabilistic formulation giving a specified confidence that the proposed procedures lead to a correct selection.

Following now the subset selection formulation of the problem (cf. for example, Gupta (1966), Gupta and Panchapakesan (1972), Chattopadhyay (1981)), a correct selection (CS_1) means selecting a nonempty subset of the k populations so that the best population associated with $H^{\varphi}_{[k]}$ is included in this subset. In a similar manner, a correct selection ($CS_{2,\zeta}$) also means selecting a nonempty subset of the k populations so that the best population, which is closest to a standard or control population, associated with $D^{\phi}_{[1],\zeta}$ is included in this subset, with ζ being *st* or *co*, respectively. Our aim is to develop a selection rule so that the subsets of the k populations contain the best populations with probabilities at least p^*_1 $(1/k < p^*_1 < 1)$ and $p^*_{2,\zeta}$ $(1/k < p^*_{2,\zeta} < 1)$, respectively. If we denote by R_1 and $R_{2,\zeta}$ the respective selection rules leading to the selection of the best of k populations and selection of the best population, which is closest to a standard or control population, respectively, then the problem mathematically is stated as follows:

$$\inf_{\Omega_1} \Pr(CS_1 | R_1) \geq p^*_1, \tag{7}$$

where $\Omega_1 = \{(H^{\varphi}_1, ..., H^{\varphi}_k) : H^{\varphi}_i \geq 0, i = 1, ..., k\}$, and

$$\inf_{\Omega_{2,\zeta}} \Pr(CS_{2,\zeta} | R_{2,\zeta}) \geq p^*_{2,\zeta}, \tag{8}$$

where $\Omega_{2,\zeta} = \{(D^{\phi}_{1,\zeta}, ..., D^{\phi}_{k,\zeta}) : D^{\phi}_{i,\zeta} \geq 0, i = 1, ..., k\}$ and ζ being *st* or *co* depending on whether Π_0 is a standard or a control population.

Then the natural selection rules are respectively:

$$R_1 \text{ Rule:} \quad \begin{array}{l}\text{The procedure for selecting the population } \Pi_{(k)} \\ \text{associated with } H^{\varphi}_{[k]} \text{ is} \\ \text{Select } \Pi_i \text{ if and only if } b\widehat{H}^{\varphi}_i \geq \widehat{H}_{\max},\ b>1,\end{array} \tag{9}$$

where $\widehat{H}_{\max} = \max\{\widehat{H}^{\varphi}_1, ..., \widehat{H}^{\varphi}_k\}$ and $b = b(k, p^*_1)$ a suitably defined constant so that (7) is satisfied. In a similar manner,

$$R_{2,\varsigma} \text{ Rule:} \quad \begin{array}{l}\text{The procedure for selecting the population } \Pi_{(1)} \\ \text{associated with } D^{\phi}_{[1],\varsigma} \text{ is} \\ \text{Select } \Pi_i \text{ if and only if } c_{\varsigma}\widehat{D}^{\phi}_{i,\varsigma} \leq \widehat{D}_{\min,\varsigma},\ c_{\varsigma}<1,\end{array} \tag{10}$$

where $\widehat{D}_{\min,\varsigma} = \min\{\widehat{D}^{\phi}_{1,\varsigma}, ..., \widehat{D}^{\phi}_{k,\varsigma}\}$ and $c_{\varsigma} = c_{\varsigma}(k, p^*_{2,\varsigma})$, with ς being *st* or *co*, is again a suitably defined constant so that (8) is satisfied. In order to derive the expression for the probabilities of correct selection and their infimum, we need the distributional properties of the random quantities appeared on them, namely, the asymptotic distributions of the estimated $\varphi-$entropies and $\phi-$divergences.

3 Asymptotical results of $\widehat{H}^{\varphi}_i$, $\widehat{D}^{\phi}_{i,st}$ and $\widehat{D}^{\phi}_{i,co}$

In the propositions that follow, the asymptotic distributions of the estimated $\widehat{H}^{\varphi}_i$, $\widehat{D}^{\phi}_{i,st}$ and $\widehat{D}^{\phi}_{i,co}$, $i = 1, ..., k$, are derived. To state the results, the following notation is necessary. Denote by $W^{(i)}_0$ and $W^{(i)}_{\varsigma}$, respectively, the vectors $W^{(i)}_0 = (W^{(i)}_{01}, ..., W^{(i)}_{0M})^t$ and $W^{(i)}_{\varsigma} = (W_{i1\varsigma}, ..., W_{iM\varsigma})^t$, $i = 1, ..., k$, ς being *st* or *co*, with elements

$$W^{(i)}_{0j} = \frac{\partial}{\partial\theta_{0j}} D^{\phi}_{i,co} = \int f'_{\theta_0(j)}\phi\left(\frac{f_{\theta_i}}{f_{\theta_0}}\right) d\mu - \int \frac{f_{\theta_i} f'_{\theta_0(j)}}{f_{\theta_0}}\phi'\left(\frac{f_{\theta_i}}{f_{\theta_0}}\right) d\mu, \tag{11}$$

for $i = 1, ..., k$, $j = 1, ..., M$, and

$$W_{ij\varsigma} = \frac{\partial}{\partial\theta_{ij}} D^{\phi}_{i,\varsigma} = \int f'_{\theta_i(j)}\phi'\left(\frac{f_{\theta_i}}{f_{\theta_0}}\right) d\mu,\ i = 1, ..., k,\ j = 1, ..., M,\ \varsigma \text{ being } st \text{ or } co, \tag{12}$$

with $f'_{\theta_\nu(j)}$ the partial derivative $f'_{\theta_\nu(j)} = (\partial/\partial\theta_{\nu j}) f_{\theta_\nu}$, $\nu = 0, i$ and $i = 1, ..., k$.

Theorem 1 *Let the densities f_{θ_i} and f_{θ_0}, $i = 1, ..., k$, and the convex function ϕ satisfying regularity conditions (R1)-(R4) and (ϕ1)-(ϕ2), respectively. Then,*

a) As $n_i \to \infty$, $i = 1, ..., k$,

$$\sqrt{n_i}\left(\widehat{H}^{\varphi}_i - H^{\varphi}_i\right) \xrightarrow{L} N(0, \sigma^2_i), \qquad i = 1, ..., k,$$

with $\sigma_i^2 = T_i^t I^{-1}(\theta_i) T_i$, *where* $T_i^t = (t_{i1}, ..., t_{iM})$, $i = 1, ..., k$, *and* $t_{ij} = \int \varphi'(f_{\theta_i}) \frac{\partial f_{\theta_i}}{\partial \theta_{ij}} d\mu$, *for* $i = 1, ..., k$, $j = 1, ..., M$.

b) As $n_i \to \infty$, $i = 1, ..., k$,

$$\sqrt{n_i}\left(\widehat{D}^{\phi}_{i,st} - D^{\phi}_{i,st}\right) \xrightarrow{L} N(0, \tau^2_{i,st}), \quad i = 1, ..., k,$$

where $\tau^2_{i,st} = (W^{(i)}_{st})^t I^{-1}(\theta_i) W^{(i)}_{st}$, *and* $W^{(i)}_{st}$, $i = 1, ..., k$, *is given by (12).*

c) If $\frac{n_0}{n_0+n_i} \to \lambda_0^{(i)} > 0$ *and* $\frac{n_i}{n_0+n_i} \to \lambda^{(i)} > 0$, *as* $n_0 \to \infty$, $n_i \to \infty$, $i = 1, ..., k$, *then we have*

$$\sqrt{n_0 + n_i}\left(\widehat{D}^{\phi}_{i,co} - D^{\phi}_{i,co}\right) \xrightarrow{L} N(0, \tau^2_{i,co}), \quad i = 1, ..., k,$$

where

$$\tau^2_{i,co} = \frac{1}{\lambda_0^{(i)}}(W_0^{(i)})^t I^{-1}(\theta_0) W_0^{(i)} + \frac{1}{\lambda^{(i)}}(W^{(i)}_{co})^t I^{-1}(\theta_i) W^{(i)}_{co}, \quad i = 1, ..., k,$$

with $W_0^{(i)}$ *and* $W^{(i)}_{co}$, $i = 1, ..., k$, *are, respectively, the vectors defined by (11) and (12). By* $I(\theta_i)$, $i = 1, ..., k$, *and* $I(\theta_0)$ *we denote the corresponding Fisher information matrices.*

Proof: a) A Taylor series expansion of $H^{\varphi}(\widehat{\theta}_i)$ around θ_i gives

$$\sqrt{n_i}\left(\widehat{H}_i^{\varphi} - H_i^{\varphi}\right) = \sqrt{n_i} T_i^t (\widehat{\theta}_i - \theta_i) + o_p(1), \; i = 1, ..., k,$$

and the results follows in view of the asymptotic normality and efficiency of the maximum likelihood estimator $\widehat{\theta}_i$ of θ_i, $i = 1, ..., k$.

b-c) A proof for b) can be obtained by an argument similar to that of a). To prove case c), a Taylor series expansion of $\widehat{D}^{\phi}_{i,co} = D^{\phi}(f_{\widehat{\theta}_i}, f_{\widehat{\theta}_0})$ around θ_i, θ_0 gives

$$\sqrt{n_0{+}n_i}\left(\widehat{D}^{\phi}_{i,co} - D^{\phi}_{i,co}\right) = \sqrt{n_0{+}n_i}(W_0^{(i)})^t(\widehat{\theta}_0 - \theta_0) + \sqrt{n_0{+}n_i}(W^{(i)}_{co})^t(\widehat{\theta}_i - \theta_i) + o_p(1), \quad (13)$$

for $i = 1, ..., k$. Taking into account that $\widehat{\theta}_0$ and $\widehat{\theta}_i$ are the maximum likelihood estimators of θ_0 and θ_i, respectively, we have that

$$\sqrt{n_0 + n_i}(\widehat{\theta}_0 - \theta_0, \widehat{\theta}_i - \theta_i) \xrightarrow{L} N(0, I_i^*), \; i = 1, ..., k, \quad (14)$$

where I_i^* is the block diagonal matrix

$$I_i^* = diag\left(\frac{1}{\lambda_0^{(i)}} I^{-1}(\theta_0) \; , \; \frac{1}{\lambda^{(i)}} I^{-1}(\theta_i)\right), \; i = 1, ..., k.$$

The result is now follows from relation (13). ■

Theorem 2 *Under the assumptions and the notation introduced in Theorem 1,*

a) The random vector $\left(\sqrt{n_1}\left(\widehat{H}_1^{\varphi} - H_1^{\varphi}\right), ..., \sqrt{n_k}\left(\widehat{H}_k^{\varphi} - H_k^{\varphi}\right)\right)$ *is asymptotically distributed as multivariate normal with mean* 0 *and covariance matrix* $diag(\sigma_1^2, ..., \sigma_k^2)$.

b) The random vector $\left(\sqrt{n_1}\left(\widehat{D}^{\phi}_{1,st} - D^{\phi}_{1,st}\right), ..., \sqrt{n_k}\left(\widehat{D}^{\phi}_{k,st} - D^{\phi}_{k,st}\right)\right)$ *is asymptotically distributed as multivariate normal with mean* 0 *and covariance matrix* $diag(\tau^2_{1,st}, ..., \tau^2_{k,st})$.

c) The joint distribution of the random vector

$$\left(\sqrt{n_0+n_1}\left(\widehat{D}^{\phi}_{1,co} - D^{\phi}_{1,co}\right), ..., \sqrt{n_0+n_k}\left(\widehat{D}^{\phi}_{k,co} - D^{\phi}_{k,co}\right)\right),$$

is asymptotically multivariate normal with mean 0 *and covariance matrix* $\Sigma = [\sigma_{ij}]_{k\times k}$, *with elements*

$$\sigma_{ij} = \begin{cases} \tau^2_{i,co}, & i=j \\ \frac{1}{\sqrt{\lambda_0^{(i)}\lambda_0^{(j)}}}(W_0^{(i)})^t I^{-1}(\theta_0) W_0^{(j)}, & i\neq j \end{cases}, \quad i,j=1,...,k.$$

Proof: We will outline the proof of c). The proof of a) and b) is immediate, in view of Theorem 1 (a), (b) and independence of the estimated $\varphi-$entropies $\widehat{H}_1^{\varphi}, ..., \widehat{H}_k^{\varphi}$ and $\phi-$divergences $\widehat{D}^{\phi}_{1,st}, ..., \widehat{D}^{\phi}_{k,st}$. To prove c), we have to show that every linear combination of $\sqrt{n_0+n_i}\left(\widehat{D}^{\phi}_{i,co} - D^{\phi}_{i,co}\right)$, $i=1,...,k$, follows a univariate normal distribution. Because of independence of the random samples considered, the right hand side of (13) corresponds to a weighted sum of independent asymptotically normal distributions and therefore every linear combination of $\sqrt{n_0+n_i}\left(\widehat{D}^{\phi}_{i,co} - D^{\phi}_{i,co}\right)$, $i=1,...,k$, is asymptotically normaly distributed. Then the joint distribution of $\sqrt{n_0+n_i}\left(\widehat{D}^{\phi}_{i,co} - D^{\phi}_{i,co}\right)$, $i=1,...,k$, is asymptotically $k-$variate normal with mean vector 0 in view of Theorem 1. The elements σ_{ij} of the covariance matrix are

$$\sigma_{ij} = Cov\left(\sqrt{n_0+n_i}\left(\widehat{D}^{\phi}_{i,co} - D^{\phi}_{i,co}\right), \sqrt{n_0+n_j}\left(\widehat{D}^{\phi}_{j,co} - D^{\phi}_{j,co}\right)\right), i,j=1,...,k.$$

After considerable algebra we obtain that

$$\sigma_{ij} = \begin{cases} \sqrt{n_0+n_i}\left[(W_0^{(i)})^t Var(\widehat{\theta}_0 - \theta_0) W_0^{(i)} + (W_{co}^{(i)})^t Var(\widehat{\theta}_i - \theta_i) W_{co}^{(i)}\right], & i=j \\ \sqrt{n_0+n_i}\sqrt{n_0+n_j}(W_0^{(i)})^t Var(\widehat{\theta}_0 - \theta_0) W_0^{(j)}, & i\neq j \end{cases},$$

for $i,j=1,...,k$. The proof is now completed in view of (14). ■

4 Probabilities of correct selection

In order to obtain the probabilities of correct selection and their infimum we need the following lemma which is proved in Menéndez et. al (2000) and it is based on the results of Curnow and Dunnett (1962). We will describe the proof for the sake of completeness.

Lemma 1 *Let* $Z_1, ..., Z_k$ *be* k *standardized normal random variables with mean* 0, *variance* 1 *and correlation coefficients* ρ_{ij} *given by* $\rho_{ij} = \sum_{s,\ell=1}^{M} \beta_{s\ell}^{(i)}\beta_{s\ell}^{(j)}$, *for* $i,j=1,...,k$

and $i \neq j$. Then

$$\Pr(Z_i < h_i,\ i=1,\dots,k) = \int_{R^{M^2}} \prod_{i=1}^{k} \Phi\left(\frac{h_i - \sum_{s,\ell=1}^{M} \beta_{s\ell}^{(i)} y_{s\ell}}{\left(1 - \sum_{s,\ell=1}^{M} \left(\beta_{s\ell}^{(i)}\right)^2\right)^{1/2}} \right) \prod_{s,\ell=1}^{M} f(y_{s\ell})dy_{s\ell},$$

where Φ and f are respectively the c.d.f. and the p.d.f. of the standard normal distribution.

Proof Consider $k+M^2$ independent standard normal variables $X_1,\dots,X_k,Y_{11},\dots,Y_{1M},\dots,$ $Y_{M1},\dots,Y_{MM}$, and define the random variables Z_i, $i=1,\dots,k$, as follows:

$$Z_i = \left[1 - \sum_{s,\ell=1}^{M} \left(\beta_{s\ell}^{(i)}\right)^2\right]^{1/2} X_i + \sum_{s,\ell=1}^{M} \beta_{s\ell}^{(i)} Y_{s\ell}, \quad i=1,\dots,k.$$

Variables Z_i, $i=1,\dots,k$, satisfy the assumptions of the lemma and the proof is completed in view of relation (1.4) of Curnow and Dunnett (1962). ■

Using the above lemma, we will obtain in the sequel the probabilities of correct selection $\Pr(CS_1|R_1)$ and $\Pr(CS_{2,\zeta}|R_{2,\zeta})$, with ζ being *st* or *co* depending on whether Π_0 is a standard or a control population.

Theorem 3 *Based on the selection rules R_1 and $R_{2,\zeta}$, given by (9) and (10), respectively, we have*

a)

$$\Pr(CS_1|R_1) = \int_{-\infty}^{+\infty} \prod_{i=1}^{k-1} \Phi\left(\frac{\sqrt{n_{(i)}}\left(bx - H_{[i]}^{\varphi}\right)}{\sigma_{(i)}} \right) f(x)dx,$$

where $\sigma_{(i)}^2 = Var(\sqrt{n_{(i)}}\widehat{H}_{(i)}^{\varphi})$, $i=1,\dots,k-1$, is obtained in Theorem 1 (a), the random variable $\widehat{H}_{(i)}^{\varphi}$ is associated with $H_{[i]}^{\varphi}$, and $n_{(i)}$, for $i=1,\dots,k$, denote the sample size from the population associated with $H_{[i]}^{\varphi}$. Φ, f are respectively the c.d.f. and the p.d.f. of the standard normal distribution and b, $b>1$, is the constant of the selection rule R_1 given by (9).

b)

$$\Pr(CS_{2,st}|R_{2,st}) = \int_{-\infty}^{+\infty} \prod_{i=2}^{k} \Phi\left(\frac{\sqrt{n_{(i)}}\left(D_{[i],st}^{\phi} - c_{st}x\right)}{\tau_{(i),st}} \right) f(x)dx,$$

with $\tau_{(i),st}^2 = Var(\sqrt{n_{(i)}}\widehat{D}_{(i),st}^{\phi})$, $i=2,\dots,k$, is obtained in Theorem 1 (b) and the random variable $\widehat{D}_{(i),st}^{\phi}$ is associated with $D_{[i],st}^{\phi}$, for $i=1,\dots,k$, while c_{st}, $c_{st}<1$, is the constant related to the selection rule $R_{2,st}$ given by (10).

c)

$$\Pr(CS_{2,co}|R_{2,co}) = \int_{-\infty}^{+\infty}\int_{R^{M^2}}\prod_{i=2}^{k}\Phi\left(\frac{D^{\phi}_{[i],co} - c_{co}x - u_{(i)}\sum_{s,\ell=1}^{M}\beta^{(i)}_{s\ell}y_{s\ell}}{u_{(i)}\left(1-\sum_{s,\ell=1}^{M}\left(\beta^{(i)}_{s\ell}\right)^2\right)^{1/2}}\right)f(x)\prod_{s,\ell=1}^{M}f(y_{s\ell})dxdy_{s\ell},$$

where $u^2_{(i)} = Var\left(\widehat{D}^{\phi}_{(i),co}\right) = \tau^2_{(i),co}/\left(n_0+n_{(i)}\right)$, $i = 2,...,k$, *with* $\tau^2_{i,co}$ *is obtained in Theorem 1 (c), and* c_{co}, $c_{co} < 1$, *is the constant of the selection rule* $R_{2,co}$ *given by (10). The random variable* $\widehat{D}^{\phi}_{(i),co}$ *is associated with* $D^{\phi}_{[i],co}$, *and* $W_0^{((i))}$ *is defined by (11), in terms of* $D^{\phi}_{[i],co}$, *for* $i = 1,...,k$, *while* $\beta^{(i)}_{s\ell}$, *are defined by* $\beta^{(i)}_{s\ell} = W^{((i))}_{0s}\left[I^{-1}_{s\ell}(\theta_0)\right]^{1/2}/\sqrt{n_0}u_{(i)}$, *with* $I^{-1}_{s\ell}(\theta_0)$ *the* $(s,\ell)-$*element of the* $M\times M$ *matrix* $I^{-1}(\theta_0)$ *and* $s,\ell = 1,...,M$, $i,j = 1,...k$ *with* $i \neq j$.

Proof: a) Taking into account relation (9),

$$\begin{aligned}\Pr(CS_1|R_1) &= \Pr(\text{Selecting }\Pi_{(k)}|R_1)\\ &= \Pr(b\widehat{H}^{\varphi}_{(k)} \geq \widehat{H}_{\max})\\ &= \Pr(b\widehat{H}^{\varphi}_{(k)} \geq \widehat{H}^{\varphi}_{(i)},\ \ \forall\ \ i = 1,...,k-1)\\ &= \Pr(\bigcap_{i=1}^{k-1}\widehat{H}^{\varphi}_{(i)} \leq b\widehat{H}^{\varphi}_{(k)})\\ &= \int_{-\infty}^{+\infty}\Pr(\bigcap_{i=1}^{k-1}\widehat{H}^{\varphi}_{(i)} \leq bx)f(x)dx.\end{aligned}$$

The last equality follows from page 44 of Johnson and Kotz (1972). But from Theorem 2 (a), $\widehat{H}^{\varphi}_{(1)},...,\widehat{H}^{\varphi}_{(k-1)}$ are independent asymptotically normal random variables with $E(\widehat{H}^{\varphi}_{(i)}) = H^{\varphi}_{[i]}$ and $Var(\widehat{H}^{\varphi}_{(i)}) = \sigma^2_{(i)}/n_{(i)}$, $i = 1,...,k-1$. Thus,

$$\Pr(CS_1|R_1) = \int_{-\infty}^{+\infty}\prod_{i=1}^{k-1}\Pr(\widehat{H}^{\varphi}_{(i)} \leq bx)f(x)dx,$$

which completes the proof of part (a).

The proof of part (b) can be obtained in a similar manner and it is omitted.

c) We have

$$\begin{aligned}\Pr(CS_{2,co}|R_{2,co}) &= \Pr(\text{Selecting }\Pi_{(1)}|R_{2,co})\\ &= \Pr(c_{co}\widehat{D}^{\phi}_{(1),co} \leq \widehat{D}_{\min,co})\\ &= \Pr(\widehat{D}^{\phi}_{(i),co} \geq c_{co}\widehat{D}^{\phi}_{(1),co},\ \ \forall\ \ i = 2,...,k)\\ &= \Pr(\bigcap_{i=2}^{k}\widehat{D}^{\phi}_{(i),co} \geq c_{co}\widehat{D}^{\phi}_{(1),co})\\ &= \int_{-\infty}^{+\infty}\Pr(\bigcap_{i=2}^{k}\widehat{D}^{\phi}_{(i),co} \geq c_{co}x)f(x)dx,\end{aligned} \qquad (15)$$

where f is the p.d.f. of the standard normal distribution. The last equality follows again from page 44 of Johnson and Kotz (1972). But taking into account Theorem 2 (c), $\widehat{D}^{\phi}_{(2),co},...,\widehat{D}^{\phi}_{(k),co}$, have jointly an asymptotic $(k-1)-$dimensional normal distribution with $E(\widehat{D}^{\phi}_{(i),co}) = D^{\phi}_{[i],co}$, $Var(\widehat{D}^{\phi}_{(i),co}) = \frac{\tau^2_{(i),co}}{n_0+n_{(i)}}$, and $Cov(\widehat{D}^{\phi}_{(i),co},\widehat{D}^{\phi}_{(j),co}) = \frac{1}{n_0}\left(W_0^{((i))}\right)^t I^{-1}(\theta_0)W_0^{((j))}$, for $i,j=2,...,k$ and $i\neq j$. In order to apply Lemma 1, let $Z_i = \frac{\widehat{D}^{\phi}_{(i),co}-D^{\phi}_{[i],co}}{u_{(i)}}$ and $h_i(x) = \frac{c_{co}x-D^{\phi}_{[i],co}}{u_{(i)}}$, with $u^2_{(i)} = Var\left(\widehat{D}^{\phi}_{(i),co}\right) = \tau^2_{(i),co}/\left(n_0+n_{(i)}\right)$ for $i = 2,...,k$. In this case $Z_2,...,Z_k$ are $k-1$ standardized normal random variables with mean 0, variance 1 and the correlation coefficient ρ_{ij} of Z_i and Z_j is of the form $\rho_{ij} = \sum_{s,\ell=1}^{M}\beta^{(i)}_{s\ell}\beta^{(j)}_{s\ell} = \left(W_0^{((i))}\right)^t I^{-1}(\theta_0)W_0^{((j))}/n_0u_{(i)}u_{(j)}$, with $\beta^{(i)}_{s\ell} = W_{0s}^{((i))}\left[I^{-1}_{s\ell}(\theta_0)\right]^{1/2}/\sqrt{n_0}u_{(i)}$, $s,\ell=1,...,M$, $i,j=1,...k$ with $i\neq j$. In this context relation (15) can be stated as

$$\begin{aligned}\Pr(CS_{2,co}|R_{2,co}) &= \int_{-\infty}^{+\infty}\Pr(Z_i\geq h_i(x), i=2,...,k)f(x)dx\\ &= \int_{-\infty}^{+\infty}\Pr(Z_i\leq -h_i(x), i=2,...,k)f(x)dx,\end{aligned}$$

because probabilities and correlations are unchanged by replacing each Z_i by $-Z_i$ and taking into account the fact that the event $\bigcap_{i=2}^{k}(Z_i\leq -h_i(x))$ is equivalent to the event $\max_{2\leq i\leq k}(Z_i+h_i(x))\leq 0$ (cf. Johnson and Kotz (1972), p. 45). The last equation leads, by using Lemma 1, to the desired result. ■

Remark 1 The above theorem helps to determine the constants b, c_{co} and c_{st} of the selection rules stated in relations (9) and (10). Indeed, based on the case (a) of the theorem we have that

$$\Pr(CS_1|R_1) = \int_{-\infty}^{+\infty}\prod_{i=1}^{k-1}\Phi\left(\frac{\sqrt{n_{(i)}}\left(bx-H^{\varphi}_{[i]}\right)}{\sigma_{(i)}}\right)f(x)dx,$$

and taking into account that $H^{\varphi}_{[1]}\leq H^{\varphi}_{[2]}\leq ...\leq H^{\varphi}_{[k]}$ and the assumption $H^{\varphi}_{[i]}\geq 0$, $i=1,...,k$, we get that

$$\Pr(CS_1|R_1) \geq \int_{-\infty}^{+\infty}\prod_{i=1}^{k-1}\Phi\left(\frac{\sqrt{n_{(i)}}\left(bx-H^{\varphi}_{[k]}\right)}{\sigma_{(i)}}\right)f(x)dx.$$

Hence, the constant b required for the definition of the selection rule (9) can be determined, in view of (7), so as to satisfy

$$\int_{-\infty}^{+\infty}\prod_{i=1}^{k-1}\Phi\left(\frac{\sqrt{n_{(i)}}\left(bx-H^{\varphi}_{[k]}\right)}{\sigma_{(i)}}\right)f(x)dx = p_1^*, \qquad (16)$$

where p_1^* is defined by (7). In a similar manner, the constants c_{st} and c_{co} of the selection rule (10), can be obtained by the solution of the following equations,

$$\int_{-\infty}^{+\infty} \prod_{i=2}^{k} \Phi\left(\frac{\sqrt{n_{(i)}}(D^{\phi}_{[1],st} - c_{st}x)}{\tau_{(i),st}}\right) f(x)dx = p^*_{2,st}, \tag{17}$$

and

$$\int_{-\infty}^{+\infty} \int_{R^{M^2}} \prod_{i=2}^{k} \Phi\left(\frac{D^{\phi}_{[1],co} - c_{co}x - u_{(i)} \sum\limits_{s,\ell=1}^{M} \beta^{(i)}_{s\ell} y_{s\ell}}{u_{(i)} \left(1 - \sum\limits_{s,\ell=1}^{M} \left(\beta^{(i)}_{s\ell}\right)^2\right)^{1/2}}\right) f(x) \prod_{s,\ell=1}^{M} f(y_{s\ell}) dx dy_{s\ell} = p^*_{2,co}. \tag{18}$$

Remark 2 The last equation can be considerably simplified if the parameters θ_0 and θ_i, $i = 1, \dots, k$, of the populations are univariate. In this case an application of $\beta^{(i)}_{s\ell} = \frac{W^{((i))}_{0s}\left[I^{-1}_{s\ell}(\theta_0)\right]^{1/2}}{\sqrt{n_0}u_{(i)}}$ for $M = 1$ gives that

$$\beta_i \equiv \beta^{(i)}_{11} = W^{((i))}_{01} \left[I^{-1}_{11}(\theta_0)\right]^{1/2} / \sqrt{n_0} u_{(i)},$$

and the correlation coefficient of $\frac{\widehat{D}^{\phi}_{(i),co} - D^{\phi}_{[i],co}}{u_{(i)}}$ and $\frac{\widehat{D}^{\phi}_{(j),co} - D^{\phi}_{[j],co}}{u_{(j)}}$ is of the form $\rho_{ij} = \beta_i \beta_j$, $i, j = 2, \dots, k$ and $i \neq j$. In this context equation (18) is simplified to

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \prod_{i=2}^{k} \Phi\left(\frac{D^{\phi}_{[1],co} - c_{co}x - u_{(i)}\beta_i y}{u_{(i)}\left(1 - \beta_i^2\right)^{1/2}}\right) f(x)f(y)dxdy = p^*_{2,co}. \tag{19}$$

5 Illustrative Examples

Example 1. (*Homoscedastic Normal Populations*)

Consider k multivariate normal $N_M(\mu_i, \Sigma)$ populations Π_i, $i = 1, \dots, k$, and denote by Π_0 the standard or control population described by $N_M(\mu_0, \Sigma)$ distribution with known variance-covariance matrix $\Sigma > 0$. Consider the Kullback-Leibler divergence between Π_i and the standard or control population Π_0. This divergence is denoted by $D_{i,st}$ or $D_{i,co}$ and can be obtained from (3), for $\phi(u) = u \log u$, $u > 0$. It can be easily seen that

$$D_{i,st} = D_{i,co} = \frac{1}{2}\left(\mu_i - \mu_0\right)^t \Sigma^{-1} \left(\mu_i - \mu_0\right), \quad i = 1, \dots, k, \tag{20}$$

i.e, a half of the Mahalanobis distance between populations Π_i and Π_0. Consider firstly the case of standard population Π_0, that is, the mean vector μ_0 is known. In this case consider a random sample of size n_i from the population $N_M(\mu_i, \Sigma)$ and denote by $\overline{\mathbf{X}}_i$, $i = 1, ..., k$, the sample mean vector. Then, in view of (20), the estimator of $D_{i,st}$ is given by

$$\widehat{D}_{i,st} = \frac{1}{2}\left(\overline{\mathbf{X}}_i - \mu_0\right)^t \Sigma^{-1}\left(\overline{\mathbf{X}}_i - \mu_0\right), \quad i = 1, ..., k. \tag{21}$$

Taking into account Theorem 1 (b), the asymptotic distribution of $\sqrt{n_i}\,\widehat{D}_{i,st}$ is univariate normal with variance $\tau^2_{i,st} = \left(W^{(i)}_{st}\right)^t I^{-1}(\mu_i) W^{(i)}_{st}$, where $I(\mu_i)$ is the Fisher information matrix associated to $N_M(\mu_i, \Sigma)$ and $W^{(i)}_{st} = \frac{\partial D_{i,st}}{\partial \mu_i}$, for $i = 1, ..., k$. We can obtain that $I(\mu_i) = \Sigma^{-1}$ and $W^{(i)}_{st} = \frac{\partial D_{i,st}}{\partial \mu_i} = \Sigma^{-1}(\mu_i - \mu_0)$, for $i = 1, ..., k$. Hence,

$$\tau^2_{i,st} = 2D_{i,st}, \quad i = 1, ..., k. \tag{22}$$

Using Theorem 3(b), relation (22) and the rank $D_{[1],st} \leq D_{[2],st} \leq ... \leq D_{[k],st}$, we have that

$$\int_{-\infty}^{+\infty} \prod_{i=2}^{k} \Phi\left(\frac{\sqrt{n_{(i)}}(D_{[1],st} - c_{st}x)}{\sqrt{2D_{[k],st}}}\right) f(x)dx = p^*_{2,st}. \tag{23}$$

This relation can be used in order to determine the constant c_{st} of the selection rule (10). If the sample sizes are equal, that is, $n_1 = n_2 = ... = n_k = n$, then (23) is simplified as follows,

$$\int_{-\infty}^{+\infty} \Phi^{k-1}\left(\frac{\sqrt{n}(D_{[1],st} - c_{st}x)}{\sqrt{2D_{[k],st}}}\right) f(x)dx = p^*_{2,st}. \tag{24}$$

Consider now the case of control population Π_0, that is, the mean vector μ_0 is unknown. In this case consider an additional random sample of size n_0 from the population $N_M(\mu_0, \Sigma)$ and denote by $\overline{\mathbf{X}}_0$, the sample mean vector. Then, in view of (20), the estimator of $D_{i,co}$ is given by

$$\widehat{D}_{i,co} = \frac{1}{2}\left(\overline{\mathbf{X}}_i - \overline{\mathbf{X}}_0\right)^t \Sigma^{-1}\left(\overline{\mathbf{X}}_i - \overline{\mathbf{X}}_0\right), \quad i = 1, ..., k. \tag{25}$$

The asymptotic distribution of $\sqrt{n_0 + n_i}\,\widehat{D}_{i,co}$ is univariate normal according to Theorem 1(c), with variance

$$\tau^2_{i,co} = 2\left(\frac{1}{\lambda^{(i)}_0} + \frac{1}{\lambda^{(i)}}\right) D_{i,co}, \quad i = 1, ..., k,$$

when $\frac{n_0}{n_0+n_i} \to \lambda^{(i)}_0 > 0$ and $\frac{n_i}{n_0+n_i} \to \lambda^{(i)} > 0$, as $n_0 \to \infty$, $n_i \to \infty$, $i = 1, ..., k$. After a little algebra, we can see that the quantities $u^2_{(i)}$ and $\beta^{(i)}_{s\ell}$ of Theorem 3(c) are given by

$$u^2_{(i)} = \frac{2(n_0 + n_{(i)})}{n_0 n_{(i)}} D_{[i],co}, \quad \text{and} \quad \beta^{(i)}_{s\ell} = \frac{W^{((i))}_{0s} \sigma^{1/2}_{s\ell}}{\sqrt{n_0}\, u_{(i)}}, \tag{26}$$

with $W_{0s}^{((i))}$ the sth element of the vector $W_0^{((i))} = \frac{\partial D_{[i],co}}{\partial \mu_0} = \Sigma^{-1}\left(\mu_0 - \mu_{(i)}\right)$ and $\sigma_{s\ell}$ the $(s,\ell)-$element of the variance-covariance matrix Σ for $s,\ell = 1,...,M$, $i = 1,...,k$ and M the dimension of multivariate normal distributions. Using Theorem 3(c), relations (26) and the order $D_{[1],co} \leq D_{[2],co} \leq ... \leq D_{[k],co}$, we have that

$$\int\limits_{R^{M^2+1}} \prod_{i=2}^{k} \Phi\left(\frac{\sqrt{n_0 n_{(i)}}\left(D_{[1],co}^{\phi} - c_{co}x - \sum\limits_{s,\ell=1}^{M} W_{0s}^{((i))}\sigma_{s\ell}^{1/2} y_{s\ell}\right)}{\left(2n_{(i)}^{*} D_{[k],co}^{\phi} - n_{(i)} \sum\limits_{s,\ell=1}^{M}\left(W_{0s}^{((i))}\sigma_{s\ell}^{1/2}\right)^2\right)^{1/2}}\right) f(x) \prod_{s,\ell=1}^{M} f(y_{s\ell})dy_{s\ell}dx = p_{2,co}^{*},$$

with $n_{(i)}^{*} = n_0 + n_{(i)}$, $i = 2,...,k$. This relation can be, at least theoretically, used in order to determine the constant c_{co} of the selection rule (10). The above equation is considerably simplified in the case of univariate normal populations.

Consider the simplest case where the populations Π_i are characterized by the univariate normal distributions $N(\mu_i, \sigma^2)$, with μ_i unknown, $i = 1,...,k$ and the control population Π_0 is $N(\mu_0, \sigma^2)$, with μ_0 unknown and σ^2 known. If f_{μ_i} and f_{μ_0} are the densities of $N(\mu_i, \sigma^2)$ and $N(\mu_0, \sigma^2)$, respectively, then

$$D_{i,co} = \int\limits_{-\infty}^{+\infty} f_{\mu_i} \log \frac{f_{\mu_i}}{f_{\mu_0}} dx = \frac{\left(\mu_i - \mu_0\right)^2}{2\sigma^2}, i = 1,...,k.$$

Thus $W_0^{(i)} = \frac{\partial D_{i,co}}{\partial \mu_0} = -\frac{\mu_i - \mu_0}{\sigma^2}$ and $I\left(\mu_0\right) = I\left(\mu_i\right) = \frac{1}{\sigma^2}$, $i = 1,...,k$. Hence, taking into account (26), we have

$$u_{(i)}^2 = \frac{\left(n_0 + n_{(i)}\right)}{n_0 n_{(i)}} \frac{\left(\mu_{(i)} - \mu_0\right)^2}{\sigma^2}, \quad \beta_i \equiv \beta_{11}^{(i)} = \frac{W_{01}^{((i))}\left[I_{11}^{-1}(\theta_0)\right]^{1/2}}{\sqrt{n_0} u_{(i)}} = -\frac{\sqrt{n_{(i)}}\left(\mu_{(i)} - \mu_0\right)}{\sqrt{n_0 + n_{(i)}}\left|\mu_{(i)} - \mu_0\right|},$$

for $i = 1,...,k$. Then, taking into account relation (19), we have that

$$\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \prod_{i=2}^{k} \Phi\left(\frac{\sigma\sqrt{n_0 n_{(i)}}\left(D_{[1],co} - c_{co}x\right) + \sqrt{n_{(i)}}\left(\mu_{(i)} - \mu_0\right) y}{\sqrt{n_0}\left|\mu_{(i)} - \mu_0\right|}\right) f(x) f(y) dx dy = p_{2,co}^{*}.$$

Various approximations of the above integral had been proposed but many mathematical packages allow also this computation.

Example 2. Remaining in the framework of the previous example, consider now the real data set used in Section 4 of Thabane and Safiul Haq (1999). These data are provided by Hoque et al. (1997) in their cardio-vascular research, devoted to a preliminary experiment on hearts of rats to determine wheather $Na^{+}-H^{+}$ exchange (NHE) inhibitor 4-isopropyl-3-methylsulfonylbezoyl-guanidine (HOE 642) protects against Lysophosphatidylcholine (LPC)-induced cardiac injuries. For details of the experiment we refer the reader to Thabane and Safiul Haq (1999) and Hoque et al. (1997). Based on Thabane and Safiul Haq (1999), in this experiment a large group of hearts of rats was divided into small groups. The first group, denoted by Π_0, represents the group of healthy hearts and is considered to be the standard population. The rest of the hearts were divided into 3 more groups, which we denote by Π_1, Π_2 and Π_3. Three different treatments, (i.e. dose level of a remedy) have applied to the hearts of the three groups Π_1, Π_2 and Π_3, respectively. At the end of the experiment, measurements of ATP, creatine phosphate (CrP) and lactate (LAC) were taken from frozen cardiac tissues. The main object is to select the best treatment relative to the standard population Π_0. We assume that random samples of size $n_i = n = 6$ are coming from each of the normal $N_3(\mu_i, \Sigma)$ populations Π_i, with known common variance-covariance matrix Σ and unknown mean vectors μ_i, $i = 1, 2, 3$. Based on the data of Table 1 of Thabane and Safiul Haq (1999), the following matrix

$$\Sigma = \begin{pmatrix} 7.9154 & 7.4655 & 0.1271 \\ 7.4655 & 24.2169 & 0.0354 \\ 0.1271 & 0.0354 & 3.5078 \end{pmatrix},$$

is the pooled sample covariance matrix obtained from the sample covariance matrices given by Thabane and Safiul Haq (1999). Denote by $\overline{\mathbf{X}}_i$ the sample mean vector obtained from a sample of size $n = 6$ from the population $N_3(\mu_i, \Sigma)$, $i = 1, 2, 3$. Let $\overline{\mathbf{X}}_1 = (12.9, 12.2, 9.1)^t$, $\overline{\mathbf{X}}_2 = (16.9, 18.6, 5.6)^t$ and $\overline{\mathbf{X}}_3 = (18.6, 15.7, 3.1)^t$. Consider also the standard population Π_0, described by $N_3(\mu_0, \Sigma)$ distribution with $\mu_0 = (26.2, 18.8, 3.1)^t$. If we have decided to use Kullback-Leibler divergence, then the selection rule leading to the selection of the best population, which is closest to the standard population Π_0, is in view of (10),

$$\text{Select } \Pi_i \text{ if and only if } c_{st}\widehat{D}_{i,st} \leq \widehat{D}_{\min,st},\ c_{st} < 1,$$

where $\widehat{D}_{\min,st} = \min\{\widehat{D}_{1,st}, \widehat{D}_{2,st}, \widehat{D}_{3,st}\}$. $\widehat{D}_{i,st} = \frac{1}{2}\left(\overline{\mathbf{X}}_i - \mu_0\right)^t \Sigma^{-1} \left(\overline{\mathbf{X}}_i - \mu_0\right)$, $i = 1, 2, 3$, has been obtained by (21) and the constant c_{st} can be determined by (24), if the unknown $D_{[i],st}$ are replaced by their estimators $\widehat{D}_{[i],st}$, $i = 1, 2, 3$. We can easily obtain that

$$\widehat{D}_{1,st} = 17.76, \quad \widehat{D}_{2,st} = 8.634, \quad \widehat{D}_{3,st} = 4.132.$$

It is obvious that

$$\widehat{D}_{[1],st} = 4.132 \leq \widehat{D}_{[2],st} = 8.634 \leq \widehat{D}_{[3],st} = 17.76.$$

Because the sample sizes are equal, $n = 6$, the constant c_{st} can be determined, in view of (24), by the solution of the equality

$$\int_{-\infty}^{+\infty} \Phi^2\left(1.6982 - 0.411 c_{st} x\right) f(x) dx = p^*_{2,st}.$$

We observe that the above integral is a decreasing function of the constant c_{st}. In the following table, the value of the constant c_{st} is determined for some values of the probability of correct selection $p^*_{2,st}$,

$p^*_{2,st}$	0.91299	0.91156	0.90867	0.90652	0.90089	0.89363	0.89158
c_{st}	0.1	0.2	0.4	0.5	0.7	0.9	0.95

If we decide to use probability of correct selection $p^*_{2,st} = 0.90089$, then the value $c_{st} = 0.7$. The selection rule (10) gives that

$$c_{st}\widehat{D}_{3,st} = 2.8924 \leq 4.132 = \widehat{D}_{\min,st},$$

and the third population Π_3 is the best population with respect to the standard population Π_0. If we consider $p^*_{2,st} = 0.90867$, then $c_{st} = 0.4$. We observe that

$$\begin{matrix} c_{st}\widehat{D}_{3,st} = 1.6528 \\ c_{st}\widehat{D}_{2,st} = 3.4536 \end{matrix} \leq 4.132 = \widehat{D}_{\min,st}.$$

Hence, populations Π_3 and Π_2 can be selected as the best populations in comparison with the standard population Π_0, with a high probability of correct selection equal to 0.90867. Hoque et al. (1997) analyzed the same real data set by comparing the means of each group with the mean of the population Π_0 for each characteristic. Their conclusion was that Π_3 was the best population for the cardio-vascular problem. Thabane and Safiul Haq (1999) considered Π_2 as the best population by using their Bayesian approach to study the problem of selecting the best population. These two methods are not accompanied their conclusions with a probabilistic statement related to the correct selection.

We have to notice at this point, that the asymptotic results of the paper are used, in this example, to analyze a small sample real data set. However, although we have presented this example as a simple illustration of the results of this paper, the conclusions are in harmony with the similar ones by Hoque et al. (1997) and Thabane and Haq (1999).

Example 3. (*Regular Exponential Populations*)

Consider k independent populations Π_i, $i = 1, ..., k$, described by the regular exponential models

$$f_{\theta_i}(x) = \exp\left\{\sum_{j=1}^{M} T_j(x)\theta_{ij} - b(\theta_i)\right\}, \quad \theta_i \in \Theta \subseteq R^M, \quad i = 1, ..., k, \tag{27}$$

with

$$b(\theta_i) = \ln \int \exp\left\{\sum_{j=1}^{M} T_j(x)\theta_{ij}\right\} d\mu, \quad i = 1, ..., k,$$

and μ a probability measure on the Euclidean space. Then, the Shannon entropy, defined by (1), for the above exponential models is given by (cf. Menendez (2000)),

$$H_i = H(\theta_i) = -\sum_{j=1}^{M} \theta_{ij}\frac{\partial b(\theta)}{\partial \theta_{ij}} + b(\theta).$$

If $\widehat{\theta}_i$ is the maximum likelihood estimator of θ_i, then the estimator $\sqrt{n_i}\widehat{H}_i$, $i = 1, ..., k$, follows, in view of Theorem 1(a), a normal distribution with variance $\sigma_i^2 = T_i^t I^{-1}(\theta_i) T_i$, where $T_i^t = (t_{i1}, ..., t_{iM})$ and

$$t_{ir} = -\sum_{j=1}^{M} \theta_{ij} \frac{\partial^2 b(\theta)}{\partial\theta_{ir}\partial\theta_{ij}}, \quad I(\theta_i) = \left[\frac{\partial^2 b(\theta)}{\partial\theta_{ir}\partial\theta_{i\ell}}\right]_{M\times M}, \quad i = 1, ..., k, \quad r, \ell = 1, ..., M.$$

These asymptotic results and relation (16) can be used in order to determine the constant b of the selection rule (9).

Consider now that k independent populations Π_i, $i = 1, ..., k$, are described by the well known gamma distribution with parameters α_i and β_i, $i = 1, ..., k$, and densities

$$f_{(\alpha_i,\beta_i)}(x) = \exp\{-\alpha_i x + \beta_i \ln x + (\beta_i + 1)\ln \alpha_i - \ln\Gamma(\beta_i + 1)\}, i = 1, ..., k.$$

These densities are immediately obtained from (27) for $b(\alpha_i, \beta_i) = -(\beta_i + 1)\ln \alpha_i - \ln\Gamma(\beta_i + 1)$, $i = 1, ..., k$, $T_1(x) = \ln x$ and $T_2(x) = -x$. The Shannon entropy of $f_{(\alpha_i,\beta_i)}(x)$ is given by

$$H(\alpha_i, \beta_i) = -\beta_i\Psi(\beta_i + 1) + (\beta_i + 1) + \ln\frac{\Gamma(\beta_i + 1)}{\alpha_i}, i = 1, ..., k,$$

where Γ and $\Psi = \Gamma'/\Gamma$ are the well known gamma and digamma functions, respectively. The maximum likelihood estimators $\widehat{\alpha}_i, \widehat{\beta}_i$ of the parameters α_i, β_i can be obtained by the solution of the equations

$$\Psi(\widehat{\alpha}_i) - \ln\widehat{\alpha}_i - \ln(\widetilde{X}_i/\overline{X}_i) = 0 \quad \text{and} \quad \widehat{\beta}_i + 1 = \overline{X}_i/\widehat{\alpha}_i, \quad i = 1, ..., k,$$

where $\overline{X}_i$ is the sample mean and $\widetilde{X}_i = \prod_{j=1}^{n} X_{ij}^{1/n}$ is the sample geometric mean based on a random sample $X_{i1}, ..., X_{in_i}$ of size n_i from the gamma distribution $f_{(\alpha_i,\beta_i)}(x)$, $i = 1, ..., k$. Applying Theorem 1(a) we have that $\sqrt{n_i}H(\widehat{\alpha}_i, \widehat{\beta}_i)$ has asymptotically normal distribution with variance (cf. Menendez (2000)),

$$\sigma_i^2 = \frac{\delta(\beta_i + 1) + (\beta_i + 1)[1 - \beta_i\delta(\beta_i + 1)]^2 + 2[1 - \beta_i\delta(\beta_i + 1)]}{(\beta_i + 1)\delta(\beta_i + 1) - 1},$$

where

$$\delta(\beta_i + 1) = \frac{\Psi'(\beta_i + 1)\Gamma(\beta_i + 1) - [\Psi(\beta_i + 1)]^2}{\Gamma(\beta_i + 1)}, \quad i = 1, ..., k.$$

The formulas obtained above and relation (16) can be used in order to use the rule (9) for the selection of the best from k independent gamma populations.

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